

Variationsprinzipien für nichtlineare Eigenwertaufgaben

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The term **nonlinear eigenvalue problem** is not used in a unique way in the literature

On the one hand: Parameter dependent nonlinear (with respect to the state variable) operator equations

$$T(\lambda, u) = 0$$

are discussed concerning

- positivity of solutions
- multiplicity of solution
- dependence of solutions on the parameter; bifurcation
- (change of) stability of solutions

In this presentation

For $\lambda \in D \subset \mathbb{C}$ let $T(\lambda)$ be a linear self-adjoint and bounded operator on a Hilbert space \mathcal{H} .

Find $\lambda \in D$ and $x \neq 0$ such that

$$T(\lambda)x = 0. \quad (1)$$

Then λ is called an eigenvalue of $T(\cdot)$, and x a corresponding eigenvector.

Nonlinear eigenproblems arise in

- dynamic/stability analysis of structures and in fluid mechanics
- regularization of total least squares problems
- vibration of sandwich plates
- electronic behavior of semiconductor hetero-structures
- vibration of fluid-solid structures
- accelerator design
- vibro-acoustics of piezoelectric/poroelastic structures
- nonlinear integrated optics

Example 1: Electronic behavior of quantum dots

Semiconductor nanostructures have attracted tremendous interest in the past few years because of their special physical properties and their potential for applications in micro- and optoelectronic devices.

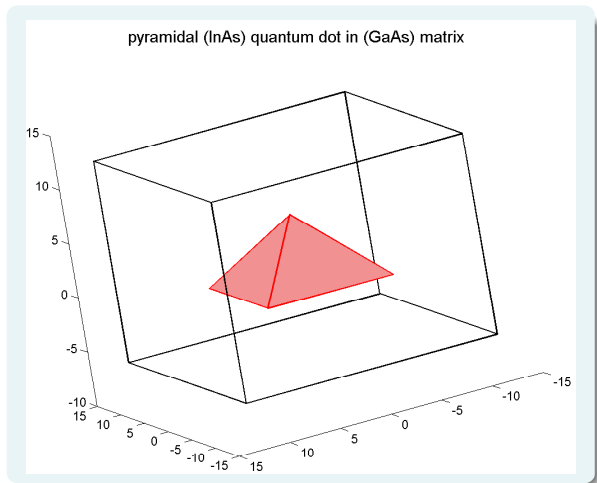
In such nanostructures, the free carriers are confined to a small region of space by potential barriers, and if the size of this region is less than the electron wavelength, the electronic states become quantized at discrete energy levels.

The ultimate limit of low dimensional structures is the quantum dot, in which the carriers are confined in all three directions, thus reducing the degrees of freedom to zero.

Therefore, a quantum dot can be thought of as an artificial atom.

Problem

Determine relevant energy states (i.e. eigenvalues) and corresponding wave functions (i.e. eigenfunctions) of a three-dimensional **quantum dot** embedded in a matrix.



Governing equation: **Schrödinger equation**

$$-\nabla \cdot \left(\frac{\hbar^2}{2m(x, E)} \nabla \Phi \right) + V(x)\Phi = E\Phi, \quad x \in \Omega_q \cup \Omega_m$$

where \hbar is the reduced Planck constant, $m(x, E)$ is the electron effective mass, and $V(x)$ is the confinement potential.

m and V are discontinuous across the heterojunction.

Boundary and interface conditions

$$\Phi = 0 \quad \text{on outer boundary of matrix } \Omega_m$$

$$\text{BenDaniel–Duke condition} \quad \frac{1}{m_m} \frac{\partial \Phi}{\partial n} \Big|_{\partial \Omega_m} = \frac{1}{m_q} \frac{\partial \Phi}{\partial n} \Big|_{\partial \Omega_q} \quad \text{on interface}$$

Find $E \in \mathbb{R}$ and $\Phi \in H_0^1(\Omega)$, $\Phi \neq 0$, $\Omega := \bar{\Omega}_q \cup \Omega_m$, such that

$$\begin{aligned} a(\Phi, \Psi; E) &:= \frac{\hbar^2}{2} \int_{\Omega_q} \frac{1}{m_q(x, E)} \nabla \Phi \cdot \nabla \Psi \, dx + \frac{\hbar^2}{2} \int_{\Omega_m} \frac{1}{m_m(x, E)} \nabla \Phi \cdot \nabla \Psi \, dx \\ &\quad + \int_{\Omega_q} V_q(x) \Phi \Psi \, dx + \int_{\Omega_m} V_m(x) \Phi \Psi \, dx \\ &= E \int_{\Omega} \Phi \Psi \, dx =: Eb(\Phi, \Psi) \quad \text{for every } \Psi \in H_0^1(\Omega) \end{aligned}$$

The dependence of $m(x, E)$ on E can be derived from the eight-band $k \cdot p$ analysis and effective mass theory. Projecting the 8×8 Hamiltonian onto the conduction band results in the single Hamiltonian eigenvalue problem with

$$m(x, E) = \begin{cases} m_q(E), & x \in \Omega_q \\ m_m(E), & x \in \Omega_m \end{cases}$$

$$\frac{1}{m_j(E)} = \frac{P_j^2}{\hbar^2} \left(\frac{2}{E + g_j - V_j} + \frac{1}{E + g_j - V_j + \delta_j} \right), \quad j \in \{m, q\}$$

where m_j is the electron effective mass, V_j the confinement potential, P_j the momentum, g_j the main energy gap, and δ_j the spin-orbit splitting in the j th region.

Other types of effective mass (taking into account the effect of strain, e.g.) appear in the literature. They are all rational functions of E where $1/m(x, E)$ is monotonically decreasing with respect to E , and that's all we need.

- $a(\cdot, \cdot, E)$ bilinear, symmetric, bounded, $H_0^1(\Omega)$ -elliptic for $E \geq 0$
- $b(\cdot, \cdot)$ bilinear, positive definite, bounded, completely continuous

By the Lax–Milgram lemma the variational eigenproblem is equivalent to

$$T(E)\phi = 0$$

where

$$T(E) : H_0^1(\Omega) \rightarrow H_0^1(\Omega), \quad E \geq 0,$$

is a family of self-adjoint and bounded operators.

Example 2: Free vibrations of fluid-solid structures

can be modelled in terms of solid displacement and fluid pressure and one obtains the classical form of an eigenproblem

$$\operatorname{div} [\sigma(u)] + \omega^2 \rho_s u = 0 \text{ in } \Omega_s,$$

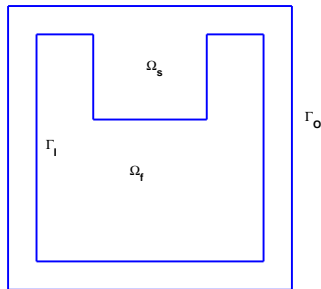
$$\Delta p + \frac{\omega^2}{c^2} p = 0 \text{ in } \Omega_f,$$

$$\sigma(u) \cdot n - pn = 0 \text{ on } \Gamma_I,$$

$$\nabla p \cdot n + \omega^2 \rho_f u \cdot n = 0 \text{ on } \Gamma_I,$$

$$u = 0 \text{ on } \Gamma_D,$$

$$\nabla p \cdot n = 0 \text{ on } \Gamma_N,$$



where

- u : solid displacement
- p : fluid pressure
- $\lambda = \omega^2$: eigenparameter
- $\sigma(u)$: linearized stress tensor
- ρ_s, ρ_f : densities of solid and fluid

Find $\lambda := \omega^2 \in \mathbb{C}$ and $(u, p) \in H_{\Gamma_D}^1(\Omega_s)^3 \times H^1(\Omega_f)$ such that

$$\begin{aligned}a_s(v, u) + c(v, p) &= \lambda b_s(v, u) \text{ and} \\ a_f(q, p) &= \lambda(-c(u, q) + b_f(q, p)).\end{aligned}$$

for every $(v, q) \in H_{\Gamma_D}^1(\Omega_s)^3 \times H^1(\Omega_f)$.

which (using the Lax-Milgram Lemma) can be transformed into a linear (but not self-adjoint) eigenvalue problem

$$K_s u + Cp = \lambda M_s u \tag{2a}$$

$$K_f p = \lambda(-C' u + M_f p) \tag{2b}$$

where $K_s : H_{\Gamma_D}^1(\Omega_s)^3 \rightarrow H_{\Gamma_D}^1(\Omega_s)^3$ is self-adjoint, elliptic, bounded, ...

Rational form of fluid-solid eigenproblem

Let $0 < \sigma_1 \leq \sigma_2 \leq \dots$ denote the eigenvalues of the decoupled eigenproblem

$$K_s u = \sigma M_s u$$

and denote by u_1, u_2, \dots corresponding orthonormal eigenfunctions. Then the spectral theorem yields

$$(K_s - \lambda M_s)^{-1} u = \sum_{n=1}^{\infty} \frac{1}{\sigma_n - \lambda} \langle u_n, u \rangle u_n.$$

If λ is not contained in the spectrum of the decoupled solid eigenproblem, then λ is an eigenvalue of the coupled fluid-solid problem if and only if it is an eigenvalue of the rational eigenvalue problem

$$T(\lambda)p := -K_f p + \lambda M_f p + \sum_{n=1}^{\infty} \frac{\lambda}{\sigma_n - \lambda} C_n p, \quad C_n p := \langle u_n, C p \rangle C' u_n.$$

$T(\lambda) : H^1(\Omega_f) \rightarrow H^1(\omega_f)$ is self-adjoint and bounded.

Quadratic form of fluid-solid eigenproblem

Another self-adjoint form of the fluid-solid eigenproblem is obtained if the second equation in (2) is multiplied by ω and p is substituted by $p =: \omega w$.

Then problem (2) is equivalent to the quadratic eigenvalue problem

$$\left(\begin{pmatrix} K_s & O \\ O & K_f \end{pmatrix} + \omega \begin{pmatrix} O & C \\ C' & O \end{pmatrix} - \omega^2 \begin{pmatrix} M_s & O \\ O & M_f \end{pmatrix} \right) \begin{pmatrix} u \\ w \end{pmatrix} = 0.$$

Variational characterization for linear eigenproblems

Let $A : \mathcal{H} \rightarrow \mathcal{H}$ a bounded linear and self-adjoint operator in a Hilbert space \mathcal{H} . Then those eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots$ above the essential spectrum of A can be characterized by three fundamental variational principles, by

- Rayleigh's principle

$$\lambda_n = \max\{R(x) : \langle x, x_i \rangle = 0, i = 1, \dots, n-1\}, \quad R(x) := \frac{\langle Ax, x \rangle}{\langle x, x \rangle}.$$

- the maxmin characterization due to Poincaré

$$\lambda_n = \max_{\dim V=n} \min_{x \in V, x \neq 0} R(x).$$

- the minmax characterization due to Courant, Fischer, and Weyl.

$$\lambda_n = \min_{\dim V=n-1} \max_{x \in V^\perp, x \neq 0} R(x).$$

These variational characterizations of eigenvalues are very powerful tools when studying self-adjoint linear operators on a Hilbert space \mathcal{H} .

Bounds for eigenvalues, comparison theorems, interlacing results and monotonicity of eigenvalues can be proved easily with these characterizations, to name just a few.

Rayleigh functional

Let

$$f : \begin{cases} J \times H & \rightarrow \mathbb{R} \\ (\lambda, x) & \mapsto \langle T(\lambda)x, x \rangle \end{cases}$$

be continuous, and assume that for every fixed $x \in \mathcal{H}$, $x \neq 0$, the real equation

$$f(\lambda, x) = 0 \tag{3}$$

has at most one solution in J .

Then equation (3) implicitly defines a functional ρ on some subset D of $\mathcal{H} \setminus \{0\}$ which we call the **Rayleigh functional**, and which is exactly the Rayleigh quotient in case of a linear eigenproblem $T(\lambda) = \lambda I - A$.

We further assume that for every $x \in D$, $x \neq 0$ and $\lambda \in J$, $\lambda \neq \rho(x)$ it holds that

$$f(\lambda, x)(\lambda - \rho(x)) > 0$$

which generalizes the definiteness of the operator B for the generalized linear eigenproblem $T(\lambda) := \lambda B - A$.

Overdamped problems

If the Rayleigh functional p is defined on the entire space $\mathcal{H} \setminus \{0\}$ then the eigenproblem $T(\lambda)x = 0$ is called **overdamped**.

This notation is motivated by the finite dimensional quadratic eigenvalue problem

$$T(\lambda)x = \lambda^2 Mx + \lambda Cx + Kx = 0$$

where M , C and K are symmetric and positive definite matrices.

- $\alpha = 0$ all eigenvalues on imaginary axis
- increase α eigenvalues go into left half plane as conjugate complex pairs
- increase α complex pairs reach real axis, run in opposite directions
- increase α all eigenvalues on the negative real axis
- increase α all eigenvalues going to the left are smaller than all eigenvalues going to the right
system is overdamped

Quadratic overdamped problems

For quadratic overdamped systems the two solutions

$$p_{\pm}(x) = \frac{1}{2} \left(-\alpha \langle Cx, x \rangle \pm \sqrt{\alpha^2 \langle Cx, x \rangle^2 - 4 \langle Mx, x \rangle \langle Kx, x \rangle} \right) / \langle Mx, x \rangle.$$

of the quadratic equation

$$\langle T(\lambda)x, x \rangle = \lambda^2 \langle Mx, x \rangle + \lambda \alpha \langle Cx, x \rangle + \langle Kx, x \rangle = 0 \quad (4)$$

are real, and they satisfy $\sup_{x \neq 0} p_{-}(x) < \inf_{x \neq 0} p_{+}(x)$.

Hence, equation (4) defines two Rayleigh functionals p_{-} and p_{+} corresponding to the intervals

$$J_{-} := \left(-\infty, \inf_{x \neq 0} p_{+}(x) \right) \quad \text{and} \quad J_{+} := \left(\sup_{x \neq 0} p_{-}(x), \infty \right).$$

For general (not necessarily quadratic) overdamped problems [Hädeler](#) (1967 for the finite dimensional case, and 1968 for $\dim \mathcal{H} = \infty$) generalized Rayleigh's principle proving that the eigenvectors are orthogonal with respect to the generalized scalar product

$$[x, y] := \begin{cases} \left\langle \frac{T(p(x)) - T(p(y))}{p(x) - p(y)} x, y \right\rangle, & \text{if } p(x) \neq p(y) \\ \langle T'(p(x))x, y \rangle, & \text{if } p(x) = p(y) \end{cases} \quad (5)$$

which is symmetric, definite and homogeneous, but in general is not bilinear.

Rayleigh's principle (Hädeler 1967, 1968)

Let $T(\lambda) : \mathcal{H} \rightarrow \mathcal{H}$, $\lambda \in J$ be a family of linear self-adjoint and bounded operators such that (1) is over-damped, and assume that for $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that $T(\lambda) + \nu(\lambda)I$ is compact.

Let $T(\cdot)$ be continuously differentiable and suppose that

$$\left. \frac{\partial}{\partial \lambda} \langle T(\lambda)x, x \rangle \right|_{\lambda=\rho(x)} > 0 \quad \text{for ever } x \neq 0.$$

Then problem $T(\lambda)x = 0$ has at most a countable set of eigenvalues in J which we assume to be ordered by magnitude $\lambda_1 \leq \lambda_2 \leq \dots$, where each eigenvalue is counted according to its multiplicity.

The corresponding eigenvectors x_1, x_2, \dots can be chosen orthonormally with respect to the generalized scalar product (5), and the eigenvalues can be determined recurrently by

$$\lambda_n = \min\{\rho(x) : [x, x_i] = 0, i = 1, \dots, n-1, x \neq 0\}.$$

Poincaré's maxmin characterization was first generalized by [Duffin](#) (1955) to overdamped quadratic eigenproblems of finite dimension, and for more general overdamped problems of [finite dimension](#) it was proved by [Rogers](#) (1964).

Infinite dimensional eigenvalue problems were studied by [Turner](#) (1967), [Langer](#) (1968), and [Weinberger](#) (1969) who proved generalizations of both, the maxmin characterization of Poincaré and of the minmax characterization of Courant, Fischer and Weyl for [quadratic](#) (and by [Turner](#) (1968) for polynomial) [overdamped](#) problems.

The corresponding generalizations for [general overdamped](#) problems of infinite dimension were derived by [Haderler](#) (1968). Similar results (weakening the compactness or smoothness requirements) are contained in [Rogers](#) (1968), [Werner](#) (1971), [Abramov](#) (1973), [Haderler](#) (1975), [Markus](#) (1985), [Maksudov & Gasanov](#) (1992), and [Hasanov](#) (2002).

Minmax and Maxmin principle (Haderer 1968)

Let $T(\lambda) : \mathcal{H} \rightarrow \mathcal{H}$, $\lambda \in J$ be a family of linear self-adjoint and bounded operators such that (1) is over-damped, and assume that for $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that $T(\lambda) + \nu(\lambda)I$ is compact.

Let $T(\cdot)$ be continuously differentiable and suppose that

$$\left. \frac{\partial}{\partial \lambda} \langle T(\lambda)x, x \rangle \right|_{\lambda=\rho(x)} > 0 \quad \text{for ever } x \neq 0.$$

Let the eigenvalues λ_n of $T(\lambda)x = 0$ be numbered in non-decreasing order regarding their multiplicities. Then they can be characterized by the following two variational principles

$$\begin{aligned} \lambda_n &= \min_{\dim V=n} \max_{x \in V, x \neq 0} \rho(x) \\ &= \max_{\dim V=n-1} \min_{x \in V^\perp, x \neq 0} \rho(x). \end{aligned}$$

Example 1: Quantum dot problem

For the quantum dot problem the family of operators $T(\lambda)$ satisfies the general conditions of the variational characterizations.

For $x \neq 0$ it holds that $a(x, x, 0) > 0$, $b(x, x) > 0$ and $\lambda \mapsto a(x, x, \lambda)$ is monotonically decreasing for $\lambda \geq 0$. Hence,

$$f(\lambda, x) = \langle T(\lambda)x, x \rangle = \lambda b(x, x) - a(x, x, \lambda) = 0$$

has exactly one positive solution $\rho(x)$, and

$$\left. \frac{\partial}{\partial \lambda} f(\lambda, x) \right|_{\lambda=\rho(x)} > 0.$$

Thus, the quantum dot problem has a countable number of non-negative eigenvalues which allow for all three variational characterizations.

Example 2: Fluid-solid vibration

For the rational eigenproblem governing free vibrations the family of operators $T(\lambda)$ satisfies the general conditions of the variational characterizations in every interval $J_n := (\sigma_{n-1}, \sigma_n)$.

$$f(\lambda, p) := \langle T(\lambda)p, p \rangle = -\langle K_f p, p \rangle + \lambda \langle M_f p, p \rangle + \sum_{n=1}^{\infty} \frac{\lambda}{\sigma_n - \lambda} |\langle u_n, Cp \rangle|^2$$

is monotonically increasing, such that $f(\lambda, p) = 0$ has at most one solution in J_n , but the Rayleigh functional is not defined on the entire space.

Non-overdamped problems

For nonoverdamped eigenproblems (i.e. $\mathcal{D}(p) \neq \mathcal{H} \setminus \{0\}$) the natural ordering to call the smallest eigenvalue the first one, the second smallest the second one, etc., is not appropriate.

This is obvious if we make a linear eigenvalue

$$T(\lambda)x := (\lambda I - A)x = 0$$

nonlinear by restricting it to an interval J which does not contain the smallest eigenvalue of A .

Then all conditions are satisfied, p is the restriction of the Rayleigh quotient R_A to $\mathcal{D}(p) := \{x \neq 0 : R_A(x) \in J\}$, and $\inf_{x \in \mathcal{D}(p)} p(x)$ will not be an eigenvalue.

Enumeration of eigenvalues

$\lambda \in J$ is an eigenvalue of $T(\cdot)$ if and only if $\mu = 0$ is an eigenvalue of the linear problem $T(\lambda)y = \mu y$. The key idea is to orientate the number of λ on the location on the eigenvalue $\mu = 0$ in the spectrum of the linear operator $T(\lambda)$.

We assume that for every $\lambda \in J$ it holds that the supremum of the essential spectrum of $T(\lambda)$ is negative (for instance: there exists $\nu(\lambda) > 0$ such that $T(\lambda) + \nu(\lambda)I$ is compact).

If $\lambda \in J$ is an eigenvalue of $T(\cdot)$ then there exists $n \in \mathbb{N}$ such that

$$0 = \max_{\dim V = n} \min_{x \in V, x \neq 0} \frac{\langle T(\lambda)x, x \rangle}{\langle x, x \rangle}.$$

In this case we assign n to the eigenvalue λ of problem $T(\lambda)x = 0$ as its number and call λ an **n -th eigenvalue of $T(\cdot)$** .

Minmax characterization (V.&B.Werner 1982, V. 2010)

Let $T(\lambda)$, $\lambda \in J$ be a family of linear self-adjoint and bounded operator on a Hilbert space \mathcal{H} depending continuously on a parameter $\lambda \in J$ where J is an open real (not necessarily bounded) interval.

Assume that

- for $x \neq 0$ $f(\lambda, x) := \langle T(\lambda)x, x \rangle = 0$ has at most one solution $p(x) \in J$, and let \mathcal{D} be the domain of the Rayleigh functional p ,
- $(\lambda - p(x))f(\lambda, x) > 0$ for every $x \in \mathcal{D}$ and $\lambda \neq p(x)$,
- and that the supremum of the essential spectrum of $T(\lambda)$ is negative for every $\lambda \in J$.

Then the nonlinear eigenvalue problem $T(\lambda)x = 0$ has at most a countable set of eigenvalues, and it holds that:

If $\lambda_n \in J$ is an n -th eigenvalue then

$$\lambda_n = \min_{\dim V=n, V \cap \mathcal{D} \neq \emptyset} \sup_{x \in \mathcal{D} \cap V} p(x). \quad (6)$$

If conversely

$$\lambda_n = \inf_{\dim V=n, V \cap \mathcal{D} \neq \emptyset} \sup_{x \in \mathcal{D} \cap V} p(x) \in J$$

then λ_n is an n -th eigenvalue of $T(\lambda)x = 0$ and (6) holds.

Sketch of proof

Step 1 (technical): Let $\lambda \in J$, and assume that V is a finite dimensional subspace of \mathcal{H} such that $V \cap \mathcal{D} \neq \emptyset$. Then it holds that

$$\lambda \left\{ \begin{array}{l} < \\ = \\ > \end{array} \right\} \sup_{x \in V \cap \mathcal{D}(\rho)} \rho(x) \quad \Leftrightarrow \quad \min_{x \in V} \langle T(\lambda)x, x \rangle \left\{ \begin{array}{l} < \\ = \\ > \end{array} \right\} 0 \quad (7)$$

Step 2: If λ_n is an n -th eigenvalue, then $\mu_n(\lambda_n) = 0$, and

$$\mu_n(\lambda_n) = \max_{\dim V=n} \min_{x \in V, \|x\|=1} \langle T(\lambda_n)x, x \rangle = \min_{x \in \tilde{V}, \|x\|=1} \langle T(\lambda_n)x, x \rangle.$$

Hence, $\min_{x \in V, \|x\|=1} \langle T(\lambda_n)x, x \rangle \leq 0$ for every V with $\dim V = n$, and (7) implies

$$\sup_{x \in V \cap \mathcal{D}} \rho(x) \geq \lambda_n = \sup_{x \in \tilde{V} \cap \mathcal{D}} \rho(x).$$

Hence, λ_n is a minmax value of ρ .

Theorem (minmax for extreme eigenvalues)

Assume that the conditions of the minmax characterization holds and that

$$\inf_{x \in \mathcal{D}} p(x) \in J.$$

If $\lambda_n \in J$ for some $n \in \mathbb{N}$ then every $V \in H_j$ with $V \cap \mathcal{D}(p) \neq \emptyset$ and $\lambda_j = \sup_{u \in V \cap \mathcal{D}(p)} p(u)$ is contained in $\mathcal{D} \cup \{0\}$, and the characterization (6) can be replaced by

$$\lambda_j = \min_{\substack{\dim V=j \\ V \subset \mathcal{D} \cup \{0\}}} \max_{v \in V, v \neq 0} p(v), \quad j = 1, \dots, n. \quad (8)$$

Maxmin characterization (V. 2003)

Assume that the conditions of the minmax characterization are satisfied.

If there is an n -th eigenvalue $\lambda_n \in J$ of $T(\lambda)x = 0$, then

$$\lambda_n = \max_{\substack{V \in \mathcal{H}_{n-1} \\ V^\perp \cap \mathcal{D} \neq \emptyset}} \inf_{v \in V^\perp \cap \mathcal{D}} \rho(v),$$

and the maximum is attained by $W := \text{span}\{u_1, \dots, u_{n-1}\}$ where u_j denotes an eigenvector corresponding to the j -largest eigenvalue $\mu_j(\lambda_n)$ of $T(\lambda_n)$.

Proof takes advantage of the following Lemma:

Let $\lambda \in J$, and let V be a finite dimensional subspace of \mathcal{H} such that $V^\perp \cap \mathcal{D} \neq \emptyset$. Then it holds that

$$\lambda \left\{ \begin{array}{l} < \\ = \\ > \end{array} \right\} \inf_{x \in V^\perp \cap \mathcal{D}(\rho)} \rho(x) \quad \Leftrightarrow \quad \max_{x \in V^\perp, \|x\|=1} \langle T(\lambda)x, x \rangle \left\{ \begin{array}{l} < \\ = \\ > \end{array} \right\} 0$$

Variational characterizations for non-overdamped nonlinear eigenvalue problems were proved (independently from our work) by

- [Barston](#) (1974) for some extreme eigenvalues of finite dimensional quadratic eigenproblems

and for the infinite dimensional case by

- [Griniv & Meln'ik](#) (1996): $T(\lambda) = A(\lambda) - I$, $A(\lambda)$ compact
- [Binding, Eschwé & H. Langer](#) (2000)
- [Eschwé & M. Langer](#) (2004) unbounded operators

Recall

$$T(\lambda)u := -K_f u + \lambda M_f u + \sum_{n=1}^{\infty} \frac{\lambda}{\sigma_n - \lambda} C_n u, \quad C_n u := \langle u_n, Cu \rangle C' u_n. \quad (9)$$

$f(\cdot, u) = \langle T(\cdot)u, u \rangle$ is strictly monotonically increasing and $f(0, u) < 0$ for every $u \neq 0$ and $\lambda \neq \sigma_\ell, \ell \in \mathbb{N}$.

Hence, in each interval $J_\ell := (\sigma_{\ell-1}, \sigma_\ell)$ (with $\sigma_0 := 0$) the eigenvalues allow for a minmax and maxmin characterization.

For $\kappa \in J_\ell$ consider the linear comparison problem

$$(K_f + \sum_{j=1}^{\ell-1} \frac{\kappa}{\kappa - \sigma_j} D_j)u = \mu(M_f + \sum_{j=\ell}^{\infty} \frac{1}{\sigma_j - \kappa} D_j)u \quad (10)$$

and denote by $R_\kappa(u)$ the corresponding Rayleigh quotient.

Let $\mu_n(\kappa)$ be the n smallest eigenvalue of the comparison problem (10).

Then $\kappa \mapsto \mu_n(\kappa)$ is monotonically increasing and λ is an n -th eigenvalue of the rational eigenproblem (9) if and only if λ is a fixed point of $\mu_n(\cdot)$.

Theorem

Let $\sigma_{\ell-1} < \kappa_1 < \kappa_2 < \sigma_\ell$ and $N(\kappa) := \max\{n \in \mathbb{N} : \mu_n(\kappa) \leq \kappa\}$.

Then the nonlinear eigenvalue problem (9) has exactly $N(\kappa_2) - N(\kappa_1)$ eigenvalues in $(\kappa_1, \kappa_2]$.

Bounds to Rayleigh functional

Lemma

Assume that $\kappa \in J_\ell$ and $R_\kappa(u) \in J_\ell$ for some $u \in \mathcal{H}$, $u \neq 0$.

Then $u \in \mathcal{D}_\ell$, and

$$\min\{\kappa, R_\kappa(u)\} \leq p_\ell(u) \leq \max\{\kappa, R_\kappa(u)\}.$$

Proof:

$$\begin{aligned} f(R_\kappa(u), u) &= \dots = \\ &= \sum_{j=1}^{\ell-1} \frac{\sigma_\ell(R_\kappa(u) - \kappa)}{(\sigma_\ell - R_\kappa(u))(\sigma_\ell - \kappa)} \langle u, D_j u \rangle + \sum_{j=\ell}^{\infty} \frac{R_\kappa(u)(R_\kappa(u) - \kappa)}{(\sigma_\ell - R_\kappa(u))(\sigma_\ell - \kappa)} \langle u, D_j u \rangle \end{aligned}$$

i.e. $f(R_\kappa(u), u) \geq 0$ for $R_\kappa(u) \geq \kappa$, and $f(R_\kappa(u), u) \leq 0$ for $R_\kappa(u) \leq \kappa$. And

$$f(\kappa, u) = \dots = (\kappa - R_\kappa(u)) \left(\sum_{j=\ell}^{\infty} \frac{1}{\sigma_j - \kappa} \langle u, D_j u \rangle + \langle u, M_f u \rangle \right),$$

i.e. $f(R_\kappa(u), u) \leq 0$ for $R_\kappa(u) \geq \kappa$, and $f(R_\kappa(u), u) \geq 0$ for $R_\kappa(u) \leq \kappa$.

Lower end of spectrum

Theorem

Assume that for $\kappa \in J_1 = (0, \sigma_1)$ the comparison problem (10) has r eigenvalues $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$ in J_1 .

Then the rational eigenvalue problem (9) has r eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ (where λ_j is a j -th eigenvalue), and it holds that

$$\min\{\mu_j, \kappa\} \leq \lambda_j \leq \max\{\mu_j, \kappa\} \quad \text{for } j = 1, \dots, n. \quad (11)$$

Proof: $\inf_{u \in \mathcal{D}_1} p_1(u) > 0$ follows by contradiction from a comparison with (10) for $\kappa = \mu_1$

The existence of λ_j , $j = 1, \dots, n$ follows from Minmax Theorem for extreme eigenvalues since the invariant subspace W of (10) corresponding to μ_n is contained in \mathcal{D}_1 and $\max_{v \in W, x \neq 0} R_\kappa(v) \in J_1$.

(11) follows from the last Lemma and the minmax Theorem for extreme eigenvalues by comparison with an invariant subspace corresponding to μ_1, \dots, μ_j .

Theorem

Assume that for $\kappa \in J_\ell$ the n -th eigenvalue of the comparison problem (10) satisfies $\mu_n(\kappa) \in (\sigma_{\ell-1}, \sigma_\ell)$. Then the rational eigenvalue problem (9) has an n -th eigenvalue $\lambda_n \in (\sigma_{\ell-1}, \sigma_\ell)$, and it holds that

$$\min\{\kappa, \mu_n(\kappa)\} \leq \lambda_n \leq \max\{\kappa, \mu_n(\kappa)\}. \quad (12)$$

Proof: Show that

- There exist V with $\dim V = n$ such that $V \cap \mathcal{D}_\ell \neq \emptyset$ and $\sup_{u \in V \cap \mathcal{D}_\ell} p_\ell(u) \leq \max\{\kappa, \mu_n(\kappa)\}$
- $\sup_{u \in V \cap \mathcal{D}_\ell} p_\ell(u) \geq \min\{\kappa, \mu_n(\kappa)\}$ for every V with $\dim V = n$ such that $V \cap \mathcal{D}_\ell \neq \emptyset$.

Then

$$\lambda_n = \inf_{\dim V=n, V \cap \mathcal{D}_\ell \neq \emptyset} \sup_{u \in V \cap \mathcal{D}_\ell} p_\ell(u) \in J_\ell,$$

i.e. λ_n is an n -th eigenvalue of (9) and inequalities (12) holds.

Unsymmetric fluid-solid eigenproblem

Recall

Find $\lambda := \omega^2 \in \mathbb{C}$ and $(u, p) \in H_{\Gamma_D}^1(\Omega_s)^3 \times H^1(\Omega_f)$ such that

$$a_s(v, u) + c(v, p) = \lambda b_s(v, u) \text{ and} \quad (13a)$$

$$a_f(q, p) = \lambda(-c(u, q) + b_f(q, p)). \quad (13b)$$

for every $(v, q) \in H_{\Gamma_D}^1(\Omega_s)^3 \times H^1(\Omega_f)$.

Then its adjoint problem is

Find $\lambda \in \mathbb{C}$ and nonzero $(u, p) \in H_{\Gamma_D}^1(\Omega_s)^3 \times H^1(\Omega_f)$ such that

$$a_s(u, v) = \lambda(b_s(u, v) - c(v, p)) \quad (14a)$$

$$c(u, q) + a_f(p, q) = \lambda b_f(p, q). \quad (14b)$$

for all $(v, q) \in H_{\Gamma_D}^1(\Omega_s)^d \times H^1(\Omega_f)$.

Lemma

- (u, p) is an eigensolution of (13) corresponding to an eigenvalue λ if and only if $(\lambda u, p)$ is an eigenvalue of the adjoint problem (14) corresponding to the same eigenvalue λ .
- Eigensolutions (u_1, p_1) and (u_2, p_2) corresponding to distinct eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal with respect to the inner product

$$\langle (u, p), (v, q) \rangle := a_s(u, v) + b_f(p, q).$$

- The eigenvalue problem (13) has only real non-negative eigenvalues.

Rayleigh functional of unsymmetric fluid-solid problem

Using the adjoint eigenfunction as a test function in equation (13) one obtains

$$\lambda a_s(u, u) + \lambda c(u, p) + a_f(p, p) = \lambda^2 b_s(u, u) - \lambda c(u, p) + \lambda b_f(p, p)$$

for any eigensolution $(\lambda, (u, p))$, i.e. λ is the positive root of

$$g(\lambda, (u, p)) := \lambda^2 b_s(u, u) + \lambda (b_f(p, p) - a_s(u, u) - 2c(u, p)) - a_f(p, p).$$

Definition: The functional $r : H_{\Gamma_D}^1(\Omega_s)^d \times H^1(\Omega_f) \setminus \{0\} \rightarrow \mathbb{R}$, where any nonzero $(u, p) \in H_{\Gamma_D}^1(\Omega_s)^d \times H^1(\Omega_f)$ is mapped to the maximal root of $g(\cdot, (u, p))$ is called the **nonlinear Rayleigh functional** of (13), i.e.

$$r(u, p) = \begin{cases} \Delta + \sqrt{\Delta^2 + \frac{a_f(p, p)}{b_s(u, u)}} & \text{if } b_s(u, u) \neq 0, \\ \frac{a_f(p, p)}{b_f(p, p)} & \text{if } b_s(u, u) = 0, \end{cases}$$

where

$$\Delta = \frac{1}{2} \frac{-b_f(p, p) + a_s(u, u) + 2c(u, p)}{b_s(u, u)}.$$

Unsymmetric fluid-solid eigenproblem

Lemma

Let $I = \mathbb{N}$ or $I = \{1, \dots, m\}$ an index set, $(u_i, p_i)_{i \in I}$ linearly independent eigenfunctions of (13) corresponding to distinct eigenvalues $(\lambda_i)_{i \in I}$ enumerated in ascending order, $\lambda_i \leq \lambda_j$ if $i < j$ and

$$(u, p) = \sum_{i \in I} (u_i, p_i).$$

(i) It holds for any $j \in I$ that

$$g(\lambda_j, (u, p)) = g(\lambda_j, (u, p) - (u_j, p_j)).$$

(ii) It holds that

$$\lambda_1 \leq r((u, p)) \leq \sup_{i \in I} \lambda_i.$$

Variational characterization

Theorem

Let $\lambda_1 \leq \lambda_2 \leq \dots$ be the eigenvalues of the unsymmetric eigenproblem (13) in ascending order and $(u_1, p_1), (u_2, p_2), \dots$ corresponding eigenfunctions. Then it holds that

(i) (Rayleigh's principle)

$$\lambda_k = \min\{r(u, p) : a_s(u, u_j) + b_f(p, p_j) = 0, j = 1, \dots, k - 1\},$$

(ii) (Min-max Characterization)

$$\lambda_k = \min_{\substack{S_k \subset H_{\Gamma_D}^1(\Omega_s)^d \times H^1(\Omega_f) \\ \dim S_k = k}} \max_{0 \neq (u, p) \in S_k} r(u, p).$$

Discretization of rational eigenproblems

Orthogonal projection of the rational eigenvalue problem describing the electronic states of a quantum dot yields a rational matrix eigenvalue problem which inherits the variational characterizations of its eigenvalues.

This enables its efficient numerical solution (V. 2006) by iterative projection methods of Nonlinear Arnoldi (V. 2003,2004) and Jacobi-Davidson type (T. Betcke & V. 2004).

Refined quantum dot models were considered including wetting layers (Markiewicz & V. 2006), spin orbit splitting (M. Betcke & V. 2007), and a magnetic fields (M. Betcke & V. 2009). The last two problems are no longer overdamped.

The same holds true for the rational eigenvalue problem governing free vibrations of fluid-solid structures. Here the additional problem arises how to initialize the method when looking for eigenvalues right to a pole (V. 2003), or restarting the iterative projection method locally if a large number of eigenpairs is sought in the interior of the spectrum (Markiewicz & V. 2005, M. Betcke & V. 2008).

Rayleigh's principle for the unsymmetric eigenproblem suggests a Rayleigh functional iteration which is cubically convergent.

This suggests efficient structure preserving iterative projection methods (Stammberger & V. 2010).