Nonlinear low rank modification of a symmetric eigenvalue problem

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Introduction

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— eigenvalue problems in fiber optic design (Kaufman 2006) where $c = e_n$

— eigenvibrations of mechanical structures with one elastically attached load (V. 2005, Solovev 2006) where $s(\lambda) = \frac{\lambda}{\sigma - \lambda}$

— more general low rank modifications arise if more than one load is attached or in vibrations of fluid–solid structures

$$Kx = \lambda Mx + \sum_{j=1}^{p} \frac{\lambda}{\sigma_j - \lambda} C_j C_j^T x, \quad C_j \in \mathbb{R}^{n \times k_j} \text{ of small rank.}$$

Outline

Constant rank 1 modification

Theorem
Let $A \in \mathbb{C}^{n\times n}$ be Hermitian with eigenvalues $\alpha_1 \ge \alpha_2 \ge \ldots \ge \alpha_n$, and let

$$B := A + \tau cc^H \quad \text{with } c \in \mathbb{C}^n \text{ and } \tau \in \mathbb{R}$$

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with eigenvalues $\beta_1 \geq \ldots \geq \beta_n$.

Then it holds that ($\alpha_{n+1} = -\infty$, $\alpha_0 = \infty$)

$$\alpha_i \leq \beta_i \leq \alpha_{i-1} \qquad \text{for} \quad \tau > 0,\ i = 1, \ldots, n \tag{1}$$

$$\alpha_{i+1} \leq \beta_i \leq \alpha_i \qquad \text{for} \quad \tau < 0,\ i = 1, \ldots, n. \tag{2}$$

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If A is diagonal with distinct diagonal entries $\alpha_n < \ldots < \alpha_1$, and if all components of c are different from zero then (1) and (2) even hold with strict inequalities.

Nonlinear rank 1 modification

Theorem

(i) For $k \in \{2, \dots, n\}$ let $\phi \in C[\alpha_k, \alpha_{k-1}]$ be nonnegative. Then the nonlinear eigenvalue problem

$$(A + \phi(\lambda)cc^H)x = \lambda x \tag{3}$$

has an eigenvalue $\hat{\lambda} \in [\alpha_k, \alpha_{k-1}]$.

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(ii) For $k \in \{1, \dots, n-1\}$ let $\phi \in C[\alpha_{k+1}, \alpha_k]$ be nonpositive. Then the nonlinear eigenvalue problem (3) has an eigenvalue $\hat{\lambda} \in [\alpha_{k+1}, \alpha_k]$.

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Proof: The kth largest eigenvalue $\mu_k(\lambda)$ of $(A + \phi(\lambda) cc^T) x = \mu x$ maps the interval $[\alpha_k, \alpha_{k-1}]$ continuously into itself.

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If *c* is an eigenvector of *A* corresponding to α_1 and *x* is an eigenvector of (3) corresponding to an eigenvalue in $[\alpha_1, \infty)$ then *x* can be shown to be a multiple of *c* and $\alpha_1 + \phi(\lambda) = \lambda$.

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Hence, for $\phi(\lambda) > \lambda - \alpha_1$ for every $\lambda \geq \alpha_1$ there does not exist an eigenvalue of (3) in $[\alpha_1, \infty)$.

Uniqueness

Theorem Assume that the conditions of part (i) of the last Theorem hold and that for some $\delta > 0$ the condition

$$\frac{\phi(\lambda) - \phi(\mu)}{\lambda - \mu} \|c\|^2 \leq 1 - \delta \tag{4}$$

is satisfied for $\lambda, \mu \in I := [\alpha_k, \alpha_{k-1}]$, $\lambda \neq \mu$. Then problem (3) has at most one eigenvalue $\hat{\lambda} \in (\alpha_k, \alpha_{k-1})$.

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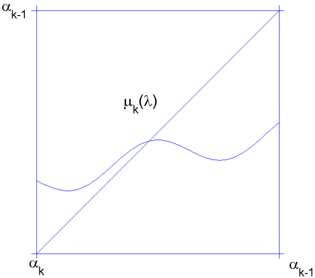

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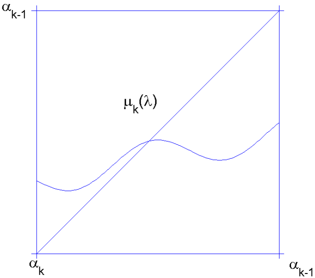

If ϕ is differentiable then (4) is equivalent to $\phi'(\lambda)\|c\|^2 \leq 1 - \delta$ for every $\lambda \in [\alpha_k, \alpha_{k-1}]$

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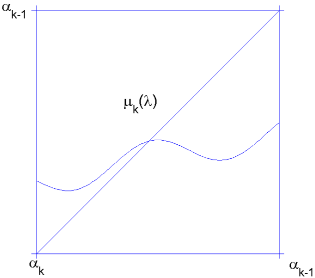

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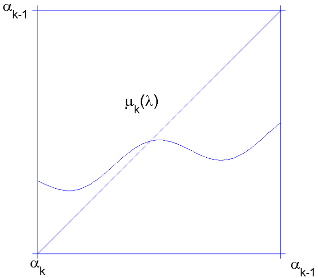

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Huang, Bai & Su assume $\phi'(\lambda) \leq 0$

A corresponding result holds for $\phi \leq 0$

Rank 1 modification

Condition (4) can not be weakened further. If c is a normalized eigenvector of A corresponding to $\tilde{\lambda}$ and $\phi(\lambda) := \lambda - \tilde{\lambda}$ then problem (3) is equivalent to the singular linear eigenvalue problem

$$(A - \tilde{\lambda} cc^T)x = \lambda (I - cc^T)x$$

for which every $\lambda \in \mathbb{R}$ is an eigenvalue with eigenvector c.

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Theorem Let $\phi \in C[\alpha_1, \infty)$ be nonnegative and assume that condition (4) is satisfied in $[\alpha_n, \infty)$. Then the nonlinear eigenvalue problem (3) has an eigenvalue $\hat{\lambda} \in [\alpha_1, \infty)$ which is the largest eigenvalue of

$$(A + \phi(\hat{\lambda})cc^T)x = \mu x.$$

Interlacing result

Theorem Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function of one sign such that

$$\frac{\phi(\lambda) - \phi(\mu)}{\lambda - \mu} \|c\|^2 \leq 1 - \delta$$

holds for all $\lambda, \mu \in \mathbb{R}$ with $\lambda \neq \mu$. Then the nonlinear eigenvalue problem

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has exactly n eigenvalues λ_k, $k = 1, \dots, n$.

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The following interlacing properties are satisfied

$$\alpha_n \leq \lambda_n \leq \alpha_{n-1} \leq \dots \leq \lambda_2 \leq \alpha_1 \leq \lambda_1 \quad \text{if } \phi(\lambda) \geq 0 \text{ for } \lambda \in \mathbb{R},$$

and

$$\lambda_n \leq \alpha_n \leq \lambda_{n-1} \leq \dots \leq \alpha_2 \leq \lambda_1 \leq \alpha_1 \quad \text{if } \phi(\lambda) \leq 0 \text{ for } \lambda \in \mathbb{R}.$$

No sign condition on ϕ

Theorem For $\phi \in C[\alpha_{k+1}, \alpha_{k-1}]$, $k = 2, \dots, n-1$ the nonlinear eigenvalue problem

$$(A + \phi(\lambda) cc^T)x = \lambda x$$

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It has at most one eigenvalue $\hat{\lambda} \in (\alpha_{k+1}, \alpha_{k-1})$ with this property if ϕ satisfies condition (4) in $[\alpha_{k+1}, \alpha_{k-1}]$.

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It has at most one eigenvalue $\hat\lambda \in (\alpha_{k+1}, \alpha_{k-1})$ with this property if ϕ satisfies condition (4) in $[\alpha_{k+1}, \alpha_{k-1}]$.

If $\phi \in C(\mathbb{R})$ satisfies condition (4) in $\mathbb{R}$, then (3) has exactly n eigenvalues $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$, and $\alpha_{k-1} \le \lambda_k \le \alpha_{k+1}$.

Rank k modification

Theorem(Weyl 1912, Parlett 1998)
Let $\beta_1 \geq \ldots \geq \beta_n$ denote the eigenvalues of $B := A + \tau H$.
Then it holds that

$$\alpha_{i+\nu} \leq \beta_i \leq \alpha_{i-\pi} \quad \text{for } \tau > 0,\ i = 1, \ldots, n,$$

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where (π, ν, ζ) denotes the inertia of H and $\alpha_j = \infty$ for $j < 1$, $\alpha_j = -\infty$ for $j > n$.

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Theorem For $k \in \{\pi + 1, \ldots, n - \nu\}$ let $\phi \in C[\alpha_{k-\nu}, \alpha_{k+\pi}]$ be nonnegative.
Then the nonlinear eigenvalue problem

$$(A + \phi(\lambda)H)x = \lambda x \tag{5}$$

has an eigenvalue $\hat{\lambda} \in [\alpha_{k+\nu}, \alpha_{k-\pi}]$. $\hat{\lambda}$ is the kth largest eigenvalue of the linear eigenvalue problem

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uniqueness

Theorem Assume that the conditions of the last Theorem hold and that for some $\delta > 0$ the condition

$$\frac{\phi(\lambda) - \phi(\mu)}{\lambda - \mu} \|H\|_2 \le 1 - \delta, \quad \lambda, \mu \in [\alpha_{k+\nu}, \alpha_{k-\pi}],\ \lambda \neq \mu$$

is satisfied in $I := [\alpha_{k+\nu}, \alpha_{k-\pi}]$, where $\|H\|_2$ is the spectral norm of H.

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Then the eigenvalue problem

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has at most one eigenvalue $\hat{\lambda} \in (\alpha_{k-\nu}, \alpha_{k+\pi})$ which is the kth largest eigenvalue of

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Fluid-solid vibrations

Actually we are interested in a small rank perturbation

$$A(\lambda) := T(\lambda) + ED(\lambda)E^H$$

of a matrix $T(\lambda)$ depending nonlinearly on λ, e.g.

$$T(\lambda) = -K + \lambda M + \sum_{j=1,\ j\neq i}^{p} \frac{\lambda}{\sigma_j - \lambda} C_j C_j^T,$$

where $E \in \mathbb{R}^{n\times k}$ and $D(\lambda) \in \mathbb{R}^{k\times k}$ is diagonal, e.g.

$$E = C_i, \quad D(\lambda) = \frac{\lambda}{\sigma_i - \lambda} I_k.$$

Minmax characterization

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Assume that the following conditions are satisfied:

(A_1) *for every fixed* $x \neq 0$ *the real equation*

$$f(\lambda; x) := x^H T(\lambda)x = 0$$

has at most one solution $\lambda =: p(x) \in J$.

Then $f(\lambda; x) = 0$ defines a functional p on some subset $\mathcal{D}(p) \subset \mathbb{C}^n$ which is called the Rayleigh functional of T on J.

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(A_2) *for every* $x \in \mathcal{D}(p)$ *and every* $\lambda \in J$ *with* $\lambda \neq p(x)$ *it holds that*

$$(\lambda - p(x))f(\lambda, x) > 0.$$

Minmax characterization cont.

If $\hat\lambda \in J$ is an eigenvalue of the nonlinear problem $T(\lambda)x = 0$, then $\mu = 0$ is an eigenvalue of the linear problem $T(\hat\lambda)x = \mu x$, and we call $\hat\lambda$ a kth eigenvalue of $T(\cdot)$, if $\mu = 0$ is the kth largest eigenvalue of $T(\hat\lambda)$.

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Theorem (V., Werner 1982, V. 2009)

Assume that the conditions (A_1) and (A_2) are satisfied.

(i) Then for every $j \in \mathbb{N}$ there exists at most one jth eigenvalue λ_j of $T(\cdot)$, and the following characterization holds:

$$\lambda_j = \min_{\substack{\dim V = j \\ V\cap\mathcal{D}(p)\neq\emptyset}} \ \sup_{v \in V\cap\mathcal{D}(p)} p(v).$$

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then λ_j is a jth eigenvalue of $T(\cdot)$.

(iii) If J contains a j and an ℓth eigenvalue with $j < \ell$, then J contains an k th eigenvalue for $k = j, \dots, \ell$ and $\lambda_j \le \lambda_{j+1} \le \cdots \le \lambda_\ell$.

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If $\lambda_j \in J$ is a jth eigenvalue of $T(\cdot)$ and $\lambda \in J$, then it holds that

$$\lambda \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} \lambda_j \quad \Longleftrightarrow \quad \mu_j(\lambda) := \max_{\dim V = j} \min_{x \in V, x \neq 0} \frac{x^H T(\lambda) x}{x^H x} \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} 0.$$

Perturbation of nonlinear problems

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Let $E \in \mathbb{C}^{n\times k}$ have full rank. Let $D : J \to \mathbb{R}^{k\times k}$ be continuous on J, denote by $(\pi(\lambda), \nu(\lambda), \zeta(\lambda))$ be the inertia of $D(\lambda)$, and set $\pi := \max_{\lambda\in J} \pi(\lambda)$ and $\nu := \max_{\lambda\in J} \nu(\lambda)$.

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If for some j the eigenvalues $\lambda_{j-\pi} \le \dots \le \lambda_{j+\nu}$ of $T(\cdot)$ are contained in J, then the perturbed eigenvalue problem

$$A(\lambda)x := (T(\lambda) + ED(\lambda)E^H)x = 0$$

has an eigenvalue $\hat\lambda \in [\lambda_{j-\pi}, \lambda_{j+\nu}]$, and $\mu = 0$ is the jth largest eigenvalue of the linear eigenproblem $A(\hat\lambda)x = \mu x$.

Proof

Let $\eta_1(\lambda) \geq \cdots \geq \eta_n(\lambda)$ and $\mu_1(\lambda) \geq \cdots \geq \mu_n(\lambda)$ denote the eigenvalues of $T(\lambda)$ and $A(\lambda)$, respectively.

Proof

Let $\eta_1(\lambda) \geq \dots \geq \eta_n(\lambda)$ and $\mu_1(\lambda) \geq \dots \geq \mu_n(\lambda)$ denote the eigenvalues of $T(\lambda)$ and $A(\lambda)$, respectively.

Then it follows from rank-k-perturbation theorem that

$$\mu_j(\lambda) \in [\eta_{j+\nu(\lambda)}(\lambda), \eta_{j-\pi(\lambda)}(\lambda)] \subset [\eta_{j+\nu}, \eta_{j-\pi}],$$

Proof

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$$\mu_j(\lambda) \in [\eta_{j+\nu(\lambda)}(\lambda), \eta_{j-\pi(\lambda)}(\lambda)] \subset [\eta_{j+\nu}, \eta_{j-\pi}],$$

and Lemma 1 implies that

$$\mu_j(\lambda_{j-\pi}) \leq \eta_{j-\pi}(\lambda_{j-\pi}) = 0 \quad \text{and} \quad \mu_j(\lambda_{j+\nu}) \geq \eta_{j+\nu}(\lambda_{j+\nu}) = 0.$$

Proof

Let $\eta_1(\lambda) \geq \cdots \geq \eta_n(\lambda)$ and $\mu_1(\lambda) \geq \cdots \geq \mu_n(\lambda)$ denote the eigenvalues of $T(\lambda)$ and $A(\lambda)$, respectively.

Then it follows from rank-k-perturbation theorem that

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Hence, $\mu_j(\cdot)$ has a root $\hat{\lambda} \in [\lambda_{j+\nu}, \lambda_{j-\pi}]$.

Rational small rank perturbation

$$A(\lambda) := T(\lambda) + \frac{\lambda}{\sigma - \lambda} CC^H, \quad C \in \mathbb{C}^{n\times k},\ rank(C) = k.$$

Let $T(\cdot)$ with eigenvalues λ_j satisfy the conditions of the minmax characterization in an open interval J with $\sigma \in J$.

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For every interval $[\lambda_{j-k}, \lambda_j] \subset J$ with $\lambda_j < \sigma$ it holds that $\pi = k$ and $\nu = 0$. Hence, there exists an eigenvalue $\hat{\lambda} \in [\lambda_{j-k}, \lambda_j]$ of $A(\cdot)$ such that $\mu = 0$ is the jth largest eigenvalue of $A(\hat{\lambda})x = \mu x$.

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Likewise, for every interval $[\lambda_j, \lambda_{j+k}] \subset J$ with $\lambda_j > \sigma$ it holds that $\pi = 0$ and $\nu = k$. Hence, there exists an eigenvalue $\hat{\lambda} \in [\lambda_j, \lambda_{j+k}]$ of $A(\cdot)$ such that $\mu = 0$ is the jth largest eigenvalue of $A(\hat{\lambda})x = \mu x$.

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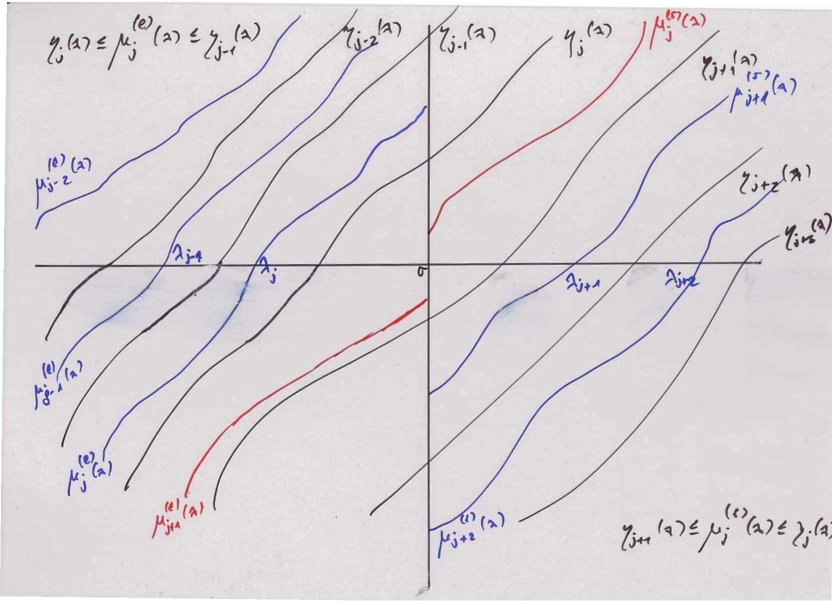
If $A(\cdot)$ satisfies the conditions of the minmax characterization then $\hat{\lambda}$ is a jth eigenvalue of $A(\cdot)$.

Close to a pole



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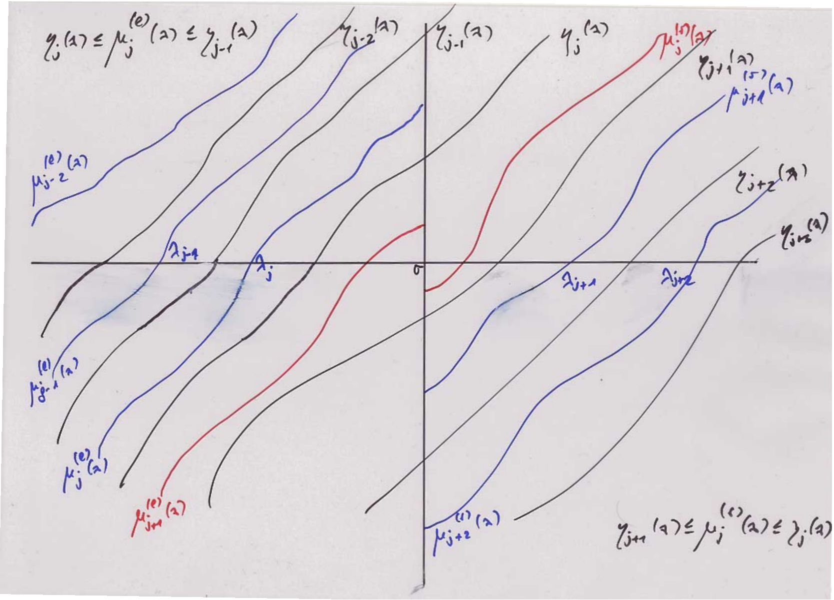

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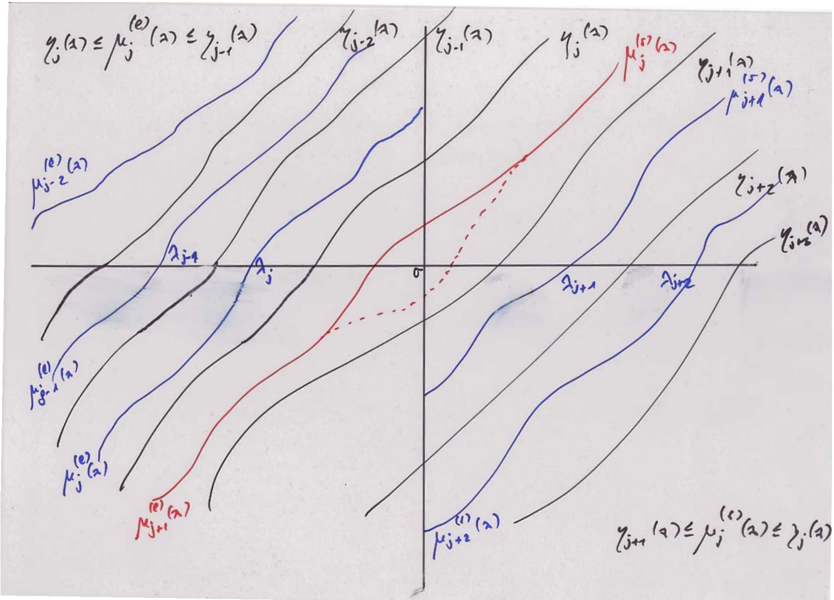

Rational small rank perturbation

Assume that $T(\cdot)$ with eigenvalues λ_j satisfies the conditions of the minmax characterization in an open interval J. Let

$$A(\lambda) := T(\lambda) + \frac{\lambda}{\sigma - \lambda} CC^H, \quad \sigma \in J,\ C \in \mathbb{C}^{n \times k},\ rank(C) = k,$$

and assume that $x^H A(\lambda) x$, $\lambda \in J$ is monotonically increasing.

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Lemma

For $\lambda > \sigma$ let $\mu_j^{(r)}(\lambda)$ be the jth largest eigenvalue of $A(\lambda)x = \mu(\lambda)x$. Then for $j = 1, \dots, n-k$ there exist the right limits

$$\nu_j := \lim_{\lambda\to\sigma+0} \mu_j^{(r)}(\lambda),$$

and ν_j is the jth largest eigenvalue of the projected eigenvalue problem

$$PT(\sigma)x = \nu x,\ C^H x = 0$$

where P denotes the orthogonal projector onto $C^\perp := \{x \in \mathbb{C}^n\ :\ C^H x = 0\}$.

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Lemma

For $\lambda < \sigma$ let $\mu_j^{(\ell)}(\lambda)$ be the jth largest eigenvalue of $A(\lambda)x = \mu(\lambda)x$. Then for $j = 1, \dots, n-k$ there exist the left limits

$$\rho_j := \lim_{\lambda\to\sigma-0} \mu_{j+k}^{(\ell)}(\lambda),$$

and ρ_j is the jth largest eigenvalue of the projected eigenvalue problem

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For continuous nonlinear small rank modifications of Hermitian eigenvalue problems we proved existence, uniqueness and interlacing results.

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These results can be generalized to continuous small rank modifications of nonlinear eigenvalue problems allowing for a minmax characterization of its eigenvalues.

The behavior of rational small rank modifications close to a pole can be fully exploited.

Remark: For the type of problems considered here the safeguarded iteration can be made a globally convergent method.