On an unsymmetric eigenvalue problem governing free vibrations of fluid-solid structures

Heinrich Voss
voss@tuhh.de

This is joint work with Markus Stammberger

Hamburg University of Technology

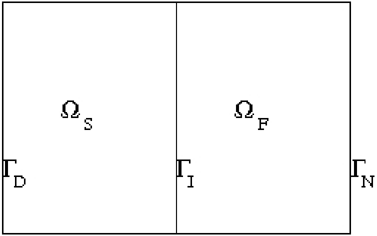
TUHH
Technische Universität Hamburg-Harburg

Outline

Problem definition

Vibrations of fluid-solid structures can be modelled in terms of solid displacement and fluid pressure and one obtains the classical form of an eigenproblem

$$\begin{aligned} \operatorname{div}\,[\sigma(u)] + \omega^2 \rho_s u &= 0 \text{ in } \Omega_s, \\ \Delta p + \frac{\omega^2}{c^2} p &= 0 \text{ in } \Omega_f, \\ \sigma(u)\cdot n - pn &= 0 \text{ on } \Gamma_I, \\ \nabla p \cdot n + \omega^2 \rho_f u \cdot n &= 0 \text{ on } \Gamma_I, \\ u &= 0 \text{ on } \Gamma_D, \\ \nabla p \cdot n &= 0 \text{ on } \Gamma_N, \end{aligned}$$



where

- u: solid displacement
- p: fluid pressure
- $\lambda = \omega^2$: eigenparameter
- $\sigma(u)$: linearized stress tensor
- ρ_s, ρ_f: densities of solid and fluid

Variational formulation

Find λ and $(u,p) \in H := (H_0^1(\Omega_s))^3 \times H_1(\Omega_f)$, $(u,p) \neq 0$ such that

$$\int_{\Omega_s} \sigma(u) : \varepsilon(v)\,dx + \int_{\Omega_f} \frac{1}{\rho_f} \nabla p \cdot \nabla q\,dx + \int_{\Gamma_I} p\,n \cdot v\,ds$$

$$= \lambda \left(\int_{\Omega_s} \rho_s uv\,dx + \int_{\Omega_f} \frac{1}{\rho_f c^2} pq\,dx - \int_{\Gamma_I} q\,u \cdot n\,ds \right)$$

for every $(v,q) \in H$.

Properties:

- There exists a countable set of eigenvalues the only cluster point of which is ∞
- All eigenvalues are real and non-negative

Discretization

Discretization by finite elements yields an unsymmetric matrix eigenproblem

$$Kx := \begin{pmatrix} K_s & C \\ 0 & K_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix} = \lambda \begin{pmatrix} M_s & 0 \\ -C^T & M_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix} =: \lambda Mx, \tag{1}$$

where

- K_s, $M_s \in \mathbb{R}^{s\times s}$ are symmetric positive definite stiffness and mass matrices of the solid,
- K_f, $M_f \in \mathbb{R}^{f\times f}$ are symmetric stiffness and mass matrices of the fluid, where K_f is positive semidefinite and M_f positive definite,
- $C \in \mathbb{R}^{s\times f}$ is due to the coupling effects between fluid and solid,
- $x_s \in \mathbb{R}^s$ is the solid displacement vector, and
- $x_f \in \mathbb{R}^f$ the fluid pressure vector.

This talks considers the properties of eigenproblem (1) and discusses ways how to use the symmetry of K_s,K_f,M_s, and M_f to adapt symmetric eigensolvers to the given problem.

Properties

- (1) can be symmetrized by $T := \begin{pmatrix} M_s^{-1}K_s & M_s^{-1}C \\ 0 & I \end{pmatrix}$, i.e.

$$T^T K x = \begin{pmatrix} K_s M_s^{-1} K_s & K_s M_s^{-1} C \\ C^T M_s^{-1} K_s & K_f + C^T M_s^{-1} C \end{pmatrix} x = \lambda \begin{pmatrix} K_s & 0 \\ 0 & M_f \end{pmatrix} x = \lambda T^T M x$$

  is a symmetric eigenvalue problem.

- (1) has only real and non-negative eigenvalues:

$$(x_s^H, x_f^H) T^T K \begin{pmatrix} x_s \\ x_f \end{pmatrix} = (K_s x_s + C x_f)^H M_s^{-1} (K_s x_s + C x_f) + x_f^H K_f x_f \geq 0$$

- If $x := \begin{pmatrix} x_s \\ x_f \end{pmatrix}$ is a right eigenvector of (1) corresponding to the eigenvalue λ, then $\hat{x} := \begin{pmatrix} \lambda x_s \\ x_f \end{pmatrix}$ is a left eigenvector.

Properties cont.

- Right eigenvectors can be chosen orthonormal with respect to

$$\tilde{M} := \begin{pmatrix} K_s & 0 \\ 0 & M_f \end{pmatrix},$$

left eigenvectors can be chosen orthogonal with respect to

$$\bar{M} := \begin{pmatrix} M_s & 0 \\ 0 & K_f \end{pmatrix}.$$

- Right eigenvectors x and left eigenvectors $\hat{x}$ corresponding to distinct eigenvalues satisfy

$$\hat{x}^T K x = \hat{x}^T M x = 0.$$

An inverse-free Rayleigh functional

A Rayleigh quotient for fluid-solid eigenproblems is given immediately by its symmetrized version.

As it involves inverse matrices it is numerically less valuable and we are interested in an inverse-free analogon.

For a given right eigenvector $\begin{pmatrix} x_s \\ x_f \end{pmatrix}$ corresponding to the eigenvalue λ it holds

$$\lambda = \frac{\begin{pmatrix} \lambda x_s \\ x_f \end{pmatrix}^T \begin{pmatrix} K_s & C \\ 0 & K_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix}}{\begin{pmatrix} \lambda x_s \\ x_f \end{pmatrix}^T \begin{pmatrix} M_s & 0 \\ -C^T & M_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix}} = \frac{\lambda x_s^T K_s x_s + \lambda x_s^T C x_f + x_f^T K_f x_f}{\lambda x_s^T M_s x_s + x_f^T M_f x_f - x_f^T C^T x_s}.$$

An inverse-free Rayleigh functional cont.

This suggests to define a Rayleigh functional p for some general $s+f$-dimensional vector by the requirement

$$p(x_s, x_f) = \frac{p(x_s, x_f)x_s^T K_s x_s + p(x_s, x_f)x_s^T C x_f + x_f^T K_f x_f}{p(x_s, x_f)x_s^T M_s x_s - x_f^T C^T x_s + x_f^T M_f x_f}$$

which leads to the quadratic equation

$$p(x_s, x_f)^2 x_s^T M_s x_s + p(x_s, x_f)(x_f^T M_f x_f - x_s^T K_s x_s - 2x_s^T C x_f) - x_f^T K_f x_f = 0.$$

We therefore choose the unique positive root of this equation as Rayleigh functional.

Rayleigh functional

Definition

$$p(x_s, x_f) := \begin{cases} q(x_s, x_f) + \sqrt{q(x_s, x_f)^2 + \frac{x_f^T K_f x_f}{x_s^T M_s x_s}} & \text{if} \quad x_s \neq 0 \\ \frac{x_f^T K_f x_f}{x_f^T M_f x_f} & \text{if} \quad x_s = 0 \end{cases} \tag{2}$$

where

$$q(x_s, x_f) := \frac{x_s^T K_s x_s - x_f^T M_f x_f + 2 x_s^T C x_f}{2 x_s^T M_s x_s}$$

is called Rayleigh functional of the fluid-solid vibration eigenvalue problem (1).

Properties

Well-known properties for symmetric eigenproblems can be generalized to the fluid-solid eigenproblem using the Rayleigh functional (2)

Differentiating the defining equation

$$p(x_s, x_f)^2 x_s^T M_s x_s + p(x_s, x_f)(x_f^T M_f x_f - x_s^T K_s x_s - 2x_s^T C x_f) - x_f^T K_f x_f = 0$$

of the Rayleigh functional yields

Lemma

Any right eigenvector x of problem (1) is a stationary point of p, i.e.

$$\nabla p(x) = 0.$$

Properties, cont.

Lemma

Assume that

$$x = \sum_{i \in I} \alpha_i x_i$$

is a linear combination of some eigenvectors indexed by I.

Then the Rayleigh functional is bounded by

$$\min_{i \in I} \lambda_i \leq p(x) \leq \max_{i \in I} \lambda_i.$$

Variational characterizations

As for symmetric eigenproblems one can derive variational characterizations of eigenvalues.

Let $\lambda_1 \leq \lambda_2 \leq \dots \lambda_{s+f}$ be the eigenvalues of problems (1). Then it holds that

Proposition (Rayleigh's principle)

$$\begin{aligned}\lambda_k &= \min\{p(x) : x^T \tilde{M} x_j = 0, j = 1, \dots, k-1\}\\ &= \max\{p(x) : x^T \tilde{M} x_j = 0, j = k+1, \dots, s+f\}\end{aligned}$$

Proposition (Minmax characterization)

$$\lambda_k = \min_{dim S_k = k} \max_{0 \neq x \in S_k} p(x) = \max_{dim S_k = s+f+1-k} \min_{0 \neq x \in S_k} p(x).$$

Structure preserving projection

Proposition

Assume that $V = \begin{pmatrix} V_s & 0 \\ 0 & V_f \end{pmatrix} \in \mathbb{R}^{s+f\times k}$ *has maximal rank* k.

Let

$$K_V := V^T K V = \begin{pmatrix} V_s^T K_s V_s & V_s^T C V_f \\ 0 & V_f^T K_f V_f \end{pmatrix}, M_V := V^T M V = \begin{pmatrix} V_s^T M_s V_s & 0 \\ -V_f^T C^T V_s & V_f^T M_f V_f \end{pmatrix};$$

and let $\tilde\lambda_1 \le \tilde\lambda_2 \le \dots \le \tilde\lambda_k$ *be the eigenvalues of the projected problem*

$$K_V y = \tilde\lambda M_V y.$$

Then it holds that

$$\lambda_j \le \tilde\lambda_j, \quad j = 1, \dots, k.$$

In particular, the jth eigenvalue λ_j does not exceed the jth eigenvalue of the structure eigenproblems $K_s x_s = \lambda M_s x_s$ and of the fluid eigenvalue problem $K_f x_f = \lambda M_f x_f$.

Rayleigh functional iteration

Require: initial vector $x^{(0)}$, $k = 0$

1: **repeat**
2: evaluate Rayleigh functional $\rho_k := p(x^{(k)})$
3: solve $(K - \rho_k M)x^{(k+1)} = Mx^{(k)}$ for $x^{(k+1)}$
4: Normalize $x^{(k+1)}$ by $\tilde{M}$
5: $k \leftarrow k + 1$
6: **until** convergence

Similar as for the Rayleigh quotient iteration for symmetric eigenproblems, local cubic convergence can be shown.

Proposition

The iterates ρ_k and $x^{(k)}$ converge locally cubically towards an eigenvalue λ and a corresponding eigenvector x.

Iterative projection methods

Rayleigh functional iteration converges fast, but is highly sensitive with respect to the given initial vector.

Moreover, for truly large problems the equation

$$(K - \rho_k M)x^{(k+1)} = Mx^{(k)} \tag{3}$$

can be solved only approximately which destroys the fast convergence.

To overcome these drawbacks, one considers iterative projection methods where the projection space V is expanded by an approximation to the direction of the Rayleigh functional iteration.

Jacobi-Davidson method

Expanding the current subspace V_k by $x^{(k+1)} := (K - \rho_k M)^{-1} M x^{(k)}$ is equivalent to expanding it by $t := x^{(k+1)} + \alpha x^{(k)}$ for every $\alpha \in \mathbb{R}$.

The most robust expansion of this type with respect to inexact solves of $(K - \rho_k M) x^{(k+1)} = M x^{(k)}$ satisfies $x^{(k)^T} \tilde{M} t = 0$ (cf. V. 2007)

which leads to the equivalent so called correction equation

$$(I - \frac{Mxx^T}{x^T M x})(K - \rho_k M)(I - \frac{xx^T \tilde{M}}{x^T \tilde{M} x}) t = -(K - \rho_k M) x, \quad t^T \tilde{M} x = 0$$

for a given eigenpair approximation (ρ_k, x).

Iterative projection methods of this type are known as Jacobi-Davidson method and were introduced by Sleijpen and van der Vorst (1996).

structure preserving Jacobi-Davidson method

The standard approach of the Jacobi-Davidson method, i.e. considering

$$V^T K V y = \lambda V^T M V y \tag{4}$$

where V is expanded by the solution of the correction equation destroys the structure of problem (1).

(4) often has non-real eigenvalues and eigenvectors.

To preserve the structure of the fluid-solid eigenproblems and to ensure real eigenvalues of the projected eigenproblem we expand the ansatz space in every step by 2 vectors $\begin{pmatrix} t_s \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ t_f \end{pmatrix}$ where $t = \begin{pmatrix} t_s \\ t_f \end{pmatrix}$ solves approximately the correction equation. Hence, we use structure preserving projection matrices

$$V = \begin{pmatrix} V_s & 0 \\ 0 & V_f \end{pmatrix}$$

and obtain

$$V^T K V = \begin{pmatrix} V_s^T K_s V_s & V_s^T C V_f \\ 0 & V_f^T K_f V_f \end{pmatrix} \text{ and } V^T M V = \begin{pmatrix} V_s^T M_s V_s & 0 \\ -V_f^T C^T V_s & V_f^T M_f V_f \end{pmatrix}.$$

Structure preserving Jacobi-Davidson method

From the minmax characterization we obtain the following monotonicity result:

Proposition

Let $\lambda_j^{(k)}$ be the j smallest eigenvalue of the kth projected eigenproblem of the structure preserving Jacobi-Davidson method.

Then it holds that

$$\lambda_j \leq \lambda_j^{(k+1)} \leq \lambda_j^{(k)} \text{ for } k = 1, \dots, \dim(V_k).$$

Structure preserving Jacobi-Davidson method

REQUIRE Initial basis $V = \text{diag}\{V_s, V_f\}$, $V_s^T K_s V_s = I$, $V_f^T M_f V_f = I$,
$m = 1$; θ_m=0;

determine preconditioner $L \approx (K - \sigma M)^{-1}$, $\sigma \approx \lambda_{\min}$
while $\theta_m \leq$ maxeig **do**
  solve the projected problem

$$\begin{pmatrix} V_s^T K_s V_s & V_s^T C V_f \\ 0 & V_f^T K_f V_f \end{pmatrix} \begin{pmatrix} y_s \\ y_f \end{pmatrix} = \theta \begin{pmatrix} V_s^T M_s V_s & 0 \\ -V_f^T C^T V_s & V_f^T M_f V_f \end{pmatrix} \begin{pmatrix} y_s \\ y_f \end{pmatrix}$$

  choose m smallest eigval. θ_m and corresp. eigvec. $(y_s^T, y_f^T)^T$
  determine Ritz vector $x = \begin{pmatrix} V_s y_s \\ V_f y_f \end{pmatrix}$ and residual $r = (K - \theta_m M)x$
  if $\|r\|/\|x\| < \epsilon$
    while $\|r\|/\|x\| < \epsilon$
      accept approximate mth eigenpair (θ_m, x); increase $m \leftarrow m + 1$;
      choose m smallest eigval. θ_m and corresp. eigvec. $(y_s^T, y_f^T)^T$
      determ. Ritz vec. $x = \begin{pmatrix} V_s y_s \\ V_f y_f \end{pmatrix}$ and res. $r = (K - \theta_m M)x$
    endwhile

reduce search space V if indicated
determine new preconditioner $L \approx (K - \theta M)^{-1}$ if necessary
endif
compute approximate solution $t = (t_s^T, t_f^T)^T$ of the correction equation

$$(I - \frac{Mxx^T}{x^T Mx})(K - \rho_m M)(I - \frac{xx^T \tilde{M}}{x^T \tilde{M} x})t = r, x^T \tilde{M} t = 0$$

orthogonalize $v_s = t_s - V_s V_s^T K_s t_s$, $v_f = t_f - V_f V_f^T M_f t_f$
if $\|v_s\|_{K_s} >$ tol expand $V_s \leftarrow [V_s, v_s/\|v_s\|_{K_s}]$
if $\|v_f\|_{M_f} >$ tol expand $V_f \leftarrow [V_f, v_f/\|v_f\|_{M_f}]$
update projected problem
endwhile

Solving correction equation

The correction equation is solved approximately by a few steps of an iterative solver (GMRES or BiCGStab).

The operator $T(\rho) := K - \rho M$ is restricted to map the subspace $x^\perp$ into itself. Hence, if $L \approx T(\sigma)^{-1}$ is a preconditioner of $T(\sigma)$, then a preconditioner for an iterativ solver of the correction equation should be modified correspondingly to

$$\tilde{L} := (I - \frac{Mxx^T}{x^T Mx})L(I - \frac{xx^T \tilde{M}}{x^T \tilde{M}x}).$$

Taking into account the projectors in the preconditioner, i.e. using $\tilde{L}$ instead of L, raises the cost of the preconditioned Krylov solver only slightly (cf. Sleijpen, van der Vorst). Only one additional linear solve with system matrix L is required.

Nonlinear Arnoldi method

Alternatively one may expand the search space by the Cayley transformation

$$t_{Ct} = (K - \sigma M)^{-1}(K - \theta M)x$$

where σ is a parameter close to the wanted eigenvalue (which is kept fixed for several iteration steps), and (θ, x) is the current approximation to the eigenpair wanted next.

Notice that it holds that

$$(\theta - \sigma)t_{ii} = (\theta - \sigma)(K - \sigma M)^{-1}Mx = x - (K - \sigma M)^{-1}(K - \theta M)x,$$

and since the current Ritz vector x is already contained in V the expansions t_{Ct} and inverse iteration

$$T_{ii} = (K - \sigma M)^{-1}Mx$$

are equivalent.

Arnoldi type iterative projection method

REQUIRE Initial basis $V = \text{diag}\{V_s, V_f\}$, $V^T V = I$, $m = 1$; θ_m=0;

determine preconditioner $L \approx (K - \sigma M)^{-1}$, $\sigma \approx \lambda_{\text{min}}$

while $\theta_m \leq$ maxeig

  solve the projected problem

$$\begin{pmatrix} V_s^T K_s V_s & V_s^T C V_f \\ 0 & V_f^T K_f V_f \end{pmatrix} \begin{pmatrix} y_s \\ y_f \end{pmatrix} = \theta \begin{pmatrix} V_s^T M_s V_s & 0 \\ -V_f^T C^T V_s & V_f^T M_f V_f \end{pmatrix} \begin{pmatrix} y_s \\ y_f \end{pmatrix}$$

  choose m smallest eigval. θ_m and corresp. eigvec. $(y_s^T, y_f^T)^T$

  determine Ritz vector $x = \begin{pmatrix} V_s y_s \\ V_f y_f \end{pmatrix}$ and residual $r = (K - \theta_m M)x$

  if $\|r\|/\|x\| < \epsilon$

    while $\|r\|/\|x\| < \epsilon$

      accept approximate mth eigenpair (θ_m, x); increase $m \leftarrow m + 1$;

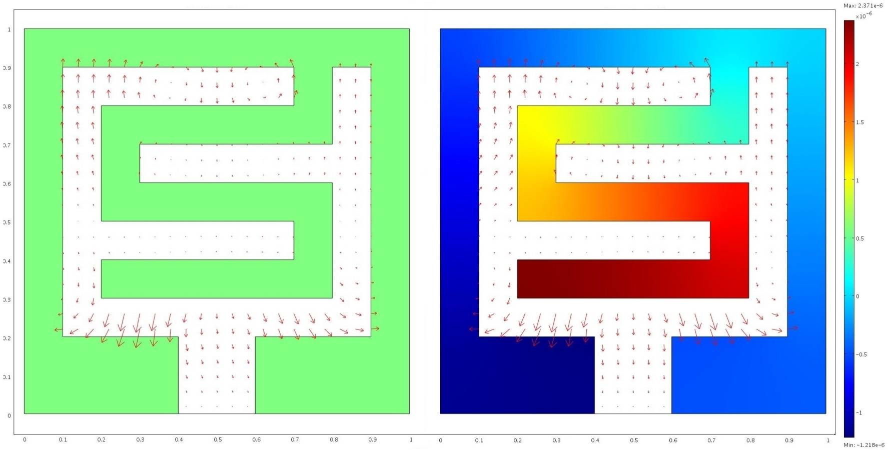
      choose m smallest eigval. θ_m and corresp. eigvec. $(y_s^T, y_f^T)^T$

      determ. Ritz vec. $x = \begin{pmatrix} V_s y_s \\ V_f y_f \end{pmatrix}$ and res. $r = (K - \theta_m M)x$

      endwhile

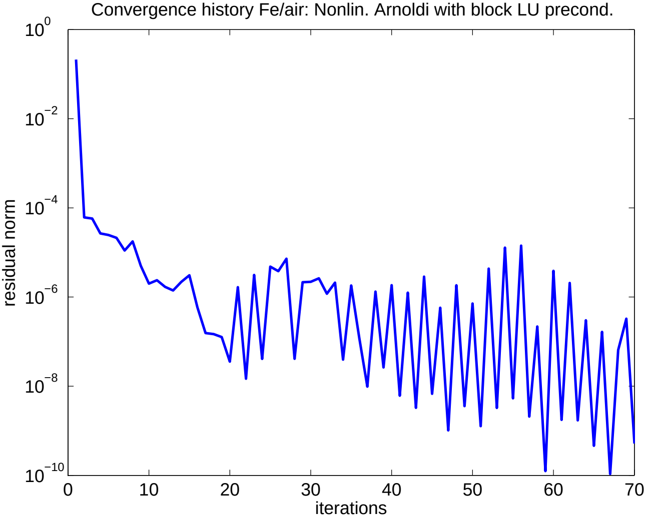
reduce search space V if indicated
determine new preconditioner $L \approx (K - \theta M)^{-1}$ if necessary
endif
solve $Lt = r$ for $t = (t_s^T, t_f^T)^T$
orthogonalise $v_s = t_s - V_s V_s^T t_s$, $v_f = t_f - V_f V_f^T t_f$
if $\|v_m\| >$ tol expand $V_m \leftarrow [V_m, v_m/\|v_m\|]$ **endif**
if $\|v_p\| >$ tol expand $V_p \leftarrow [V_p, v_p/\|v_p\|]$ **endif**
update projected problem
endwhile

Numerical Example



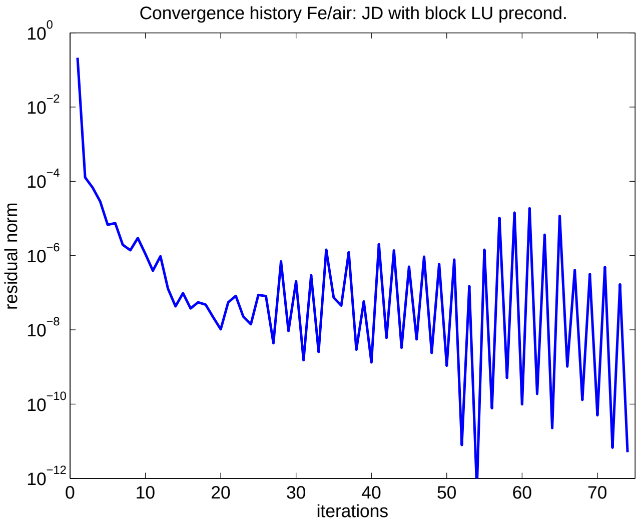
Consider a finite element model of a coupled steel/air system with 120473 DoFs (67616 Al, 52857 air).
Determine all 23 eigenfrequencies less than 500 Hz.

Numerical Example; Fe-air



CPU time (Pentium D, 3.4 GHz, 4GB) : 19.5 sec.

Numerical Example; Fe-air



CPU time : 77.9 sec.

Numerical Example; Fe-air

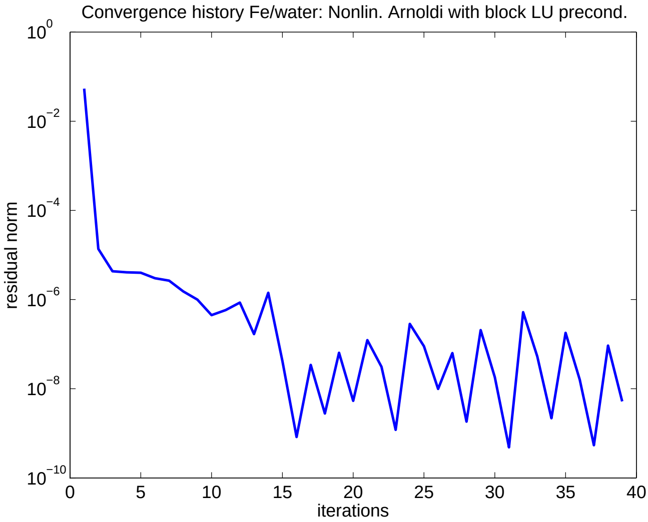
coupled	steel	air	rel.err. %
0.0000		0.0000	
41.2492		41.3165	−0.16
48.6703		48.7084	−0.08
56.9586	56.8957		0.11
75.5540	75.5076		0.06
93.1835		93.1929	−0.01
129.9855		130.0446	−0.04
150.9411	151.0261		−0.06
158.1658		158.1835	−0.01
186.4457	186.6640		−0.11
206.6432		206.3688	0.13
225.5279	225.5368		−0.01
228.3356		228.1377	0.09
265.4442		265.4166	0.01
309.8093		309.7450	0.02
334.8581		334.8499	0.01
369.7817		369.7221	0.02
407.9334		407.9114	0.01
430.8615		430.8700	−0.01
451.6980	451.7603		−0.01
453.4445		453.5362	−0.02
472.2388	472.4496		−0.01
485.7640		485.5500	0.04

Numerical Example; Fe-air

coupled	steel	air	rel.err. %	proj.	rel.err. %
0.0000		0.0000		0.0000	
41.2492		41.3165	−0.16	41.2493	$2.4e-4$
48.6703		48.7084	−0.08	48.6704	$2.0e-4$
56.9586	56.8957		0.11	56.9598	$2.2e-3$
75.5540	75.5076		0.06	75.5565	$3.3e-3$
93.1835		93.1929	−0.01	93.1836	$1.0e-4$
129.9855		130.0446	−0.04	129.9863	$6.0e-4$
150.9411	151.0261		−0.06	150.9464	$3.5e-3$
158.1658		158.1835	−0.01	158.1661	$1.6e-4$
186.4457	186.6640		−0.11	186.4536	$4.2e-3$
206.6432		206.3688	0.13	206.6441	$4.7e-4$
225.5279	225.5368		−0.01	225.5384	$4.6e-3$
228.3356		228.1377	0.09	228.3347	$4.8e-4$
265.4442		265.4166	0.01	265.4449	$2.6e-4$
309.8093		309.7450	0.02	309.8099	$2.0e-4$
334.8581		334.8499	0.01	334.8587	$1.9e-4$
369.7817		369.7221	0.02	369.7824	$1.7e-4$
407.9334		407.9114	0.01	407.9343	$2.3e-4$
430.8615		430.8700	−0.01	430.8619	$9.8e-5$
451.6980	451.7603		−0.01	451.7121	$3.1e-3$
453.4445		453.5362	−0.02	453.4464	$4.2e-4$
472.2388	472.4496		−0.01	472.3985	$2.2e-3$
485.7640		485.5500	0.04	485.7656	$3.4e-4$

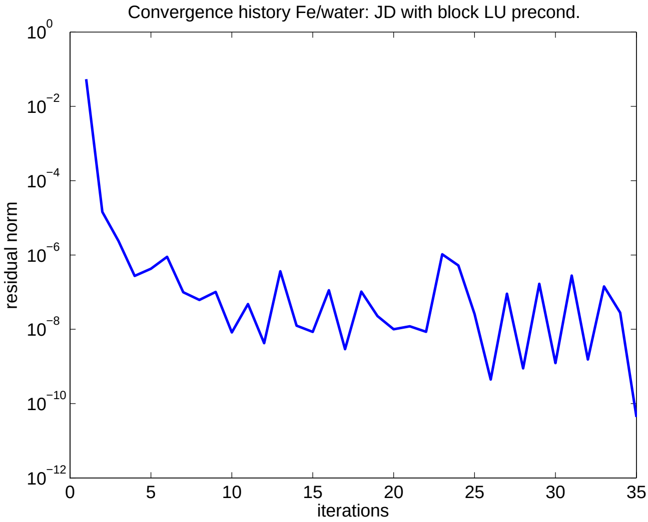
Projection to all eigenforms of fluid and structure corresponding to eigenfrequencies $\leq$ 1000 Hz.

Numerical Example; Fe-H_2O



CPU time : 10.4 sec.

Numerical Example; Fe-H_2O



CPU time : 30.0 sec.

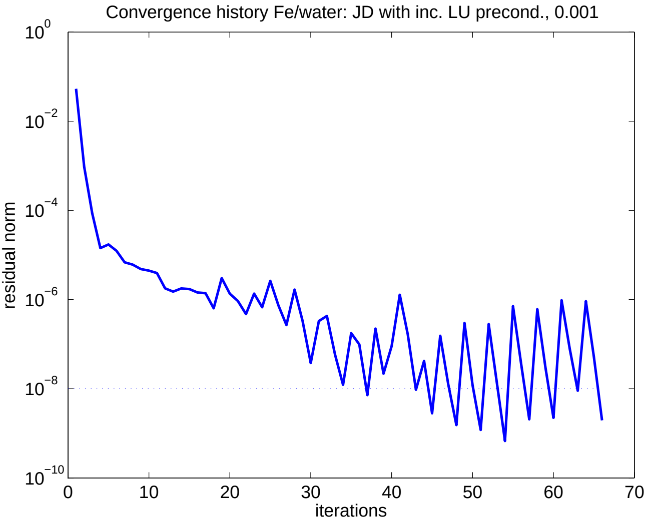
Numerical Example; Fe-H_2O

coupled	steel	water
0.0000	56.8957	0.0000
28.0148	75.5076	178.7571
41.5428	151.0261	210.7383
92.7278	186.6640	403.2021
124.7031	225.5368	
138.2591	451.7603	
270.3956	472.4496	
321.7854		
388.7314		
402.7669		

proj.	rel. err. %
0.0000	
28.1312	0.42
41.8670	0.78
93.6236	0.97
126.6132	1.53
140.4699	1.60
273.0387	0.98
325.4151	1.13
396.8814	2.10
409.8698	1.76

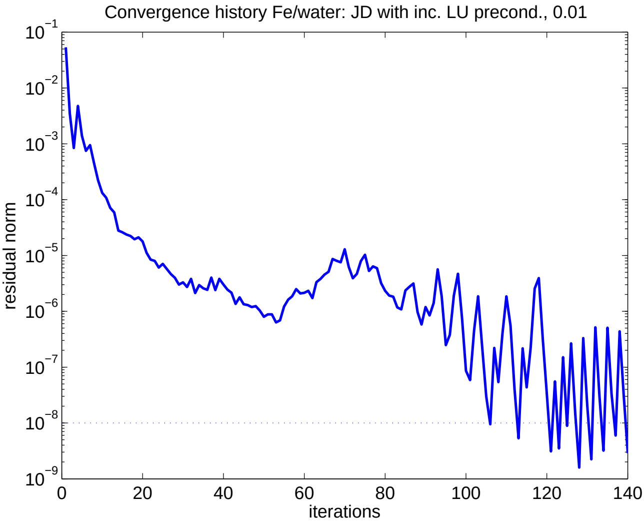
Projection to eigenforms corr. to eigenfr. $\leq$ 5000 Hz.

Numerical Example; Fe-H_2O



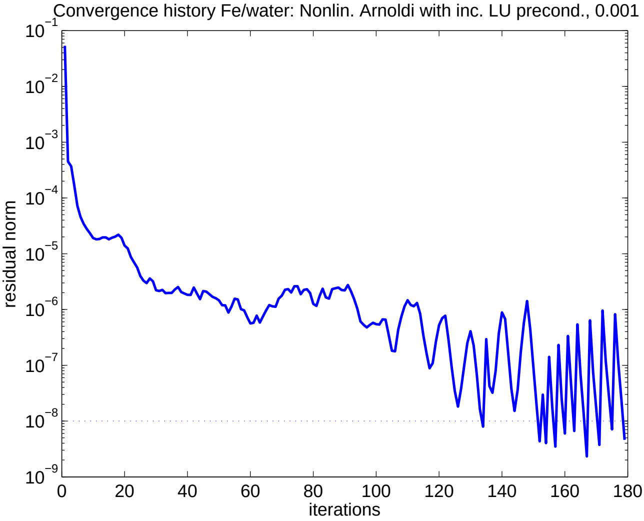
CPU time : 60.5 sec.

Numerical Example; Fe-H_2O



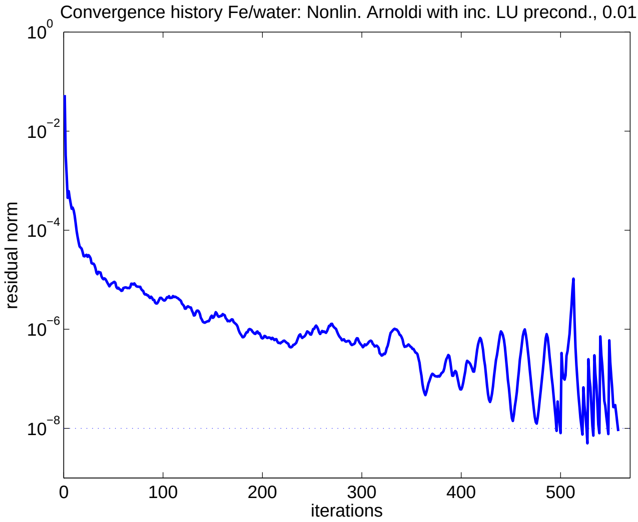
CPU time : 125.6 sec.

Numerical Example; Fe-H_2O



CPU time : 113.4 sec.

Numerical Example; Fe-H_2O



CPU time : 3411.1 sec.

Automated Multi-Level Substructring

AMLS was introduced by Bennighof (1998) and was applied to huge problems of frequency response analysis.

The large finite element model is recursively divided into very many substructures on several levels based on the sparsity structure of the system matrices.

Assuming that the interior degrees of freedom of substructures depend quasistatically on the interface degrees of freedom, and modeling the deviation from quasistatic dependence in terms of a small number of selected substructure eigenmodes the size of the finite element model is reduced substantially yet yielding satisfactory accuracy over a wide frequency range of interest.

Recent studies in vibro-acoustic analysis of passenger car bodies where very large FE models with more than six million degrees of freedom appear and several hundreds of eigenfrequencies and eigenmodes are needed have shown that AMLS is considerably faster than Lanczos type approaches.

Usual approach in automotive industry

Solve the uncoupled eigenvalue problem

$$K_u x := \begin{pmatrix} K_s & 0 \\ 0 & K_f \end{pmatrix} \begin{pmatrix} v_s \\ v_f \end{pmatrix} = \lambda \begin{pmatrix} M_s & 0 \\ 0 & M_f \end{pmatrix} \begin{pmatrix} v_s \\ v_f \end{pmatrix} =: \lambda M_u x,$$

and project the original problem to $\text{diag}\{\Phi_s, \Phi_f\}$, where the columns of Φ_s and Φ_f are eigenmodes corresponding to eigenvalues not exceeding a cut-off frequency.

Eigenpairs of the projected problem are sufficient approximations to exact eigenpairs, if the coupling is weak (fluid is air, structure is metal, e.g.)

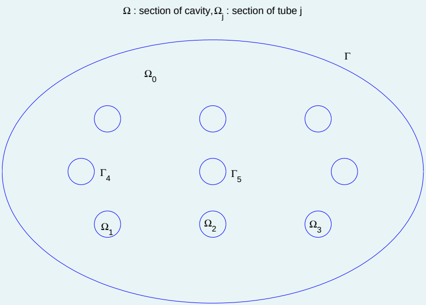
For strongly coupled problems some eigenpairs are inexact.

Example: Conca et al. 1995

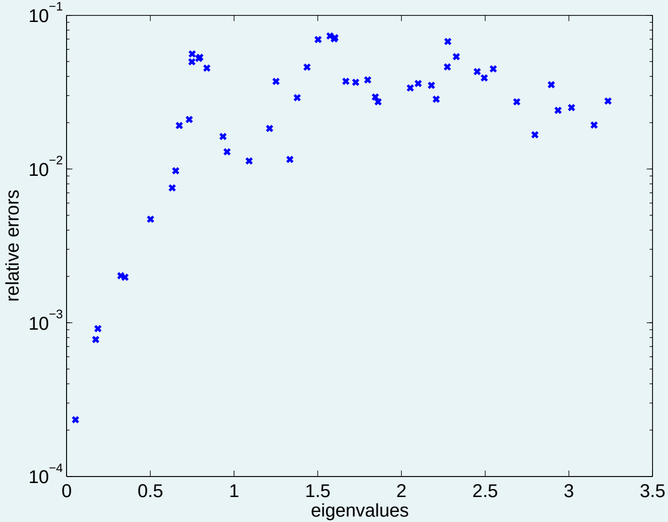
Determine acoustic eigenfrequencies of a (very long) cavity containing a tube bundle. Tubes are

- immersed in an inviscid compressible fluid
- rigid, assembled in parallel inside the fluid,
- elastically mounted such that they can vibrate transversally, but can not move in the direction perpendicular to their sections.

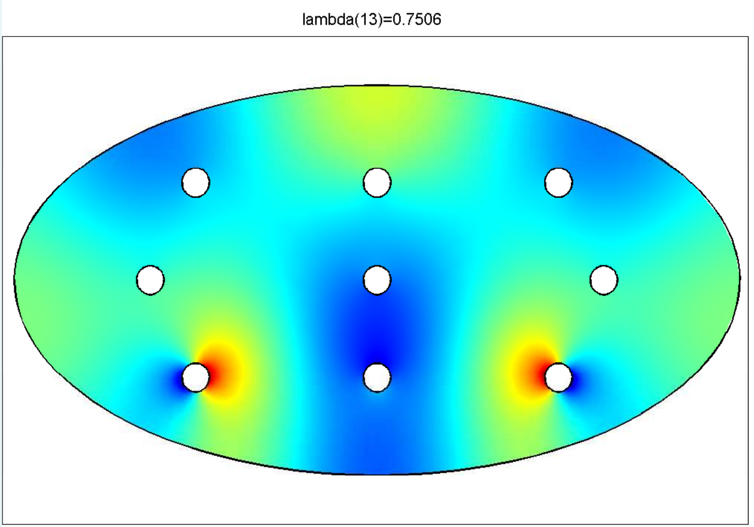
Due to these assumptions, three-dimensional effects are neglected, and the problem is studied in any transversal section of the cavity.



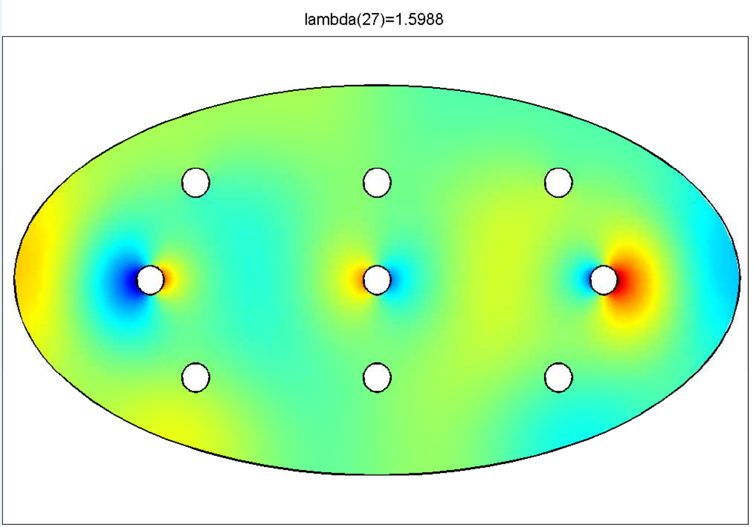
Finite element model: 143081 DoFs



Typical eigenvector



Typical eigenvector



Structure preserving AMLS

Substructure the joint graph of

$$K := \begin{pmatrix} K_s & C \\ 0 & K_f \end{pmatrix} \quad \text{and} \quad M := \begin{pmatrix} M_s & 0 \\ -C^T & M_f \end{pmatrix}$$

substructured recursively on several levels, where the coupling matrix is incorporated into the substructuring process. Hence, any substructure may consist solely of degrees of freedom from the fluid or from the solid, or caused by the coupling matrix it may contain degrees of freedom of either type.

The crucial point is to modify the AMLS algorithm such that the structure is preserved for the reduced problem, and all eigenvalues are still real.

Pure fluid or solid substructures obviously can be treated in the same way as in the symmetric AMLS method, i.e. they can be decoupled in the stiffness matrix by Gaussian elimination and reduced by modal truncation. We now describe a typical general reduction step.

Structure preserving AMLS

After a couple of reduction steps one arrives at the following matrices where the unknowns have been reordered appropriately

$$\begin{pmatrix} K_{pp} & 0 & K^f_{p\ell} & 0 & K^f_{pi} \\ K^s_{\ell p} & K^s_{\ell\ell} & C_\ell & K^s_{\ell i} K^s_{\ell i} & C_{\ell i} C_{\ell i} \\ 0 & 0 & K^f_{\ell\ell} & 0 & K^f_{\ell i} K^f_{\ell i} \\ K^s_{ip} & K^s_{i\ell} & C_{i\ell} & K^s_{ii} & C_i \\ 0 & 0 & K^f_{i\ell} & 0 & K^f_{ii} \end{pmatrix}, \begin{pmatrix} K_{pp} & M^s_{p\ell} & M^f_{p\ell} & M^s_{pi} & M^f_{pi} \\ M^s_{\ell p} & M^s_{\ell\ell} & 0 & M^s_{\ell i} & 0 \\ M^f_{\ell p} & -C^T_\ell & M^f_{\ell\ell} & -C^T_{i\ell} & M^f_{\ell i} \\ M^s_{ip} & M^s_{i\ell} & 0 & M^s_{ii} & 0 \\ M^f_{ip} & -C^T_{\ell i} & M^f_{i\ell} & -C^T_i & M^f_{ii} \end{pmatrix}. \quad (5)$$

Here p denotes the degrees of freedom obtained in the reduction steps on previous levels, ℓ collects the degrees of freedom to be handled in the current step, i corresponds to the index set of parent substructures.

Next one annihilates $K^s_{\ell i}$ and $K^f_{\ell i}$ by symmetric block Gaussian elimination, and ...

Structure preserving AMLS

solving the substructure eigenvalue problem

$$K_\ell \begin{pmatrix} \phi \\ \psi \end{pmatrix} := \begin{pmatrix} K^s_{\ell\ell} & C_\ell \\ 0 & K^f_{\ell\ell} \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \omega \begin{pmatrix} M^s_{\ell\ell} & 0 \\ -C_\ell^T & M^f_{\ell\ell} \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} =: M_\ell \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

and normalizing the eigenvectors such that

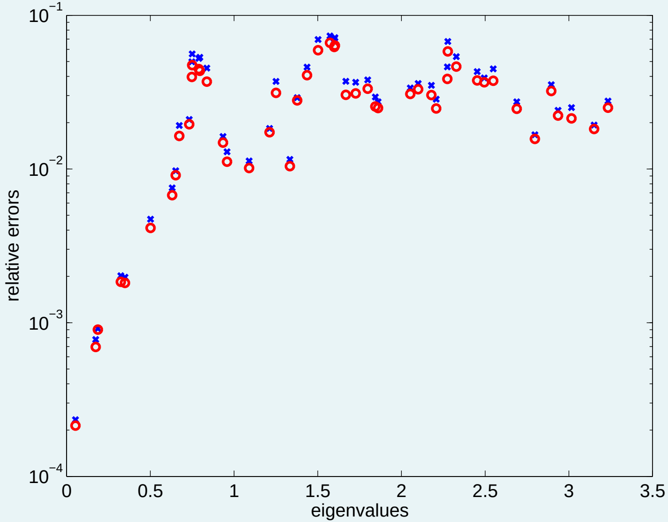
$$\left(\phi_i^T, \frac{1}{\omega_i}\psi_i^T\right) \begin{pmatrix} M^s_{\ell\ell} & 0 \\ -C_\ell^T & M^f_{\ell\ell} \end{pmatrix} \begin{pmatrix} \phi_j \\ \psi_j \end{pmatrix} = \delta_{ij}$$

the oblique projection

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & \Phi_\ell^T & \Omega^{-1}\Psi_\ell^T & 0 \\ 0 & 0 & 0 & I \end{pmatrix} (K_\ell - \lambda M_\ell) \begin{pmatrix} I & 0 & 0 \\ 0 & \Phi_\ell & 0 \\ 0 & \Psi_\ell & 0 \\ 0 & 0 & I \end{pmatrix} = 0$$

preserves the structure of (5).

Structure preserving AMLS



AMLS for fluid-solid vibrations

Consider the symmetric eigenproblem

$$\left[\begin{pmatrix} 0 & C & K_s & 0 \\ C^T & 0 & 0 & K_f \\ K_s & 0 & 0 & 0 \\ 0 & K_f & 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} M_s & 0 & 0 & 0 \\ 0 & M_f & 0 & 0 \\ 0 & 0 & K_s & 0 \\ 0 & 0 & 0 & K_f \end{pmatrix}\right] x = 0$$

If $(\lambda^2, (x_s^T, x_f^T)^T)$ is an eigenpair of the fluid-solid-problem then

$$(\pm\lambda, (\lambda^2 x_s^T \quad \pm\lambda x_f^T \quad \pm\lambda x_s^T \quad x_f^T)^T)$$

are solutions of the symmetric problem above unless $\lambda = 0$.

Partitioning based on the union of the sparsity structures of the matrices K and M gives an $(s+f)$-dimensional partitioning which can be expanded to an $2(s+f)$-dimensional partitioning so that for $i = 1, \dots, s+f$ the ith and $(i+s+f)$th degree of freedom belong to the same substructure or interface.

Standard AMLS can be performed at nearly the same cost as AMLS for an $(s+f)$ dimensional problem.

AMLS for fluid-solid vibrations