

On an unsymmetric eigenvalue problem governing free vibrations of fluid-solid structures

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- 1 Problem definition
- 2 An (inverse-free) Rayleigh functional
- 3 Structure preserving iterative projection methods
- 4 Numerical Results
- 5 AMLS

Outline

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Problem definition

Consider free vibrations of an elastic structure completely filled with a homogeneous, inviscid, and compressible fluid neglecting gravity effects. Modelling them in terms of the solid displacement and the fluid pressure, one obtains the classical form of an eigenproblem

$$\operatorname{div} [\sigma(u)] + \omega^2 \rho_s u = 0 \text{ in } \Omega_s,$$

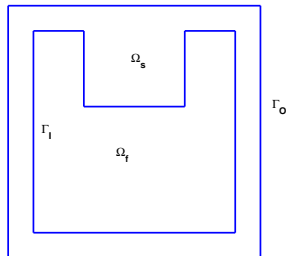
$$\Delta p + \frac{\omega^2}{c^2} p = 0 \text{ in } \Omega_f,$$

$$\sigma(u) \cdot n - pn = 0 \text{ on } \Gamma_I,$$

$$\nabla p \cdot n + \omega^2 \rho_f u \cdot n = 0 \text{ on } \Gamma_I,$$

$$u = 0 \text{ on } \Gamma_D,$$

$$\nabla p \cdot n = 0 \text{ on } \Gamma_N,$$



where

- u : solid displacement
- p : fluid pressure
- $\lambda = \omega^2$: eigenparameter
- $\sigma(u)$: linearized stress tensor

Variational formulation

Find λ and $(u, p) \in H := (H_0^1(\Omega_s))^3 \times H_1(\Omega_f)$, $(u, p) \neq 0$ such that

$$\begin{aligned} & \int_{\Omega_s} \sigma(u) : \varepsilon(v) \, dx + \int_{\Omega_f} \frac{1}{\rho_f} \nabla p \cdot \nabla q \, dx + \int_{\Gamma_I} p n \cdot v \, ds \\ &= \lambda \left(\int_{\Omega_s} \rho_s u v \, dx + \int_{\Omega_f} \frac{1}{\rho_f c^2} p q \, dx - \int_{\Gamma_I} q u \cdot n \, ds \right) \end{aligned}$$

for every $(v, q) \in H$.

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for every $(v, q) \in H$.

Properties:

- There exists a countable set of eigenvalues the only cluster point of which is ∞
- All eigenvalues are real and non-negative

Discretization

Discretization by finite elements yields an unsymmetric matrix eigenproblem

$$Kx := \begin{pmatrix} K_s & C \\ 0 & K_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix} = \lambda \begin{pmatrix} M_s & 0 \\ -C^T & M_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix} =: \lambda Mx, \quad (1)$$

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where

- $K_s, M_s \in \mathbb{R}^{s \times s}$ are symmetric positive definite stiffness and mass matrices of the solid,
- $K_f, M_f \in \mathbb{R}^{f \times f}$ are symmetric stiffness and mass matrices of the fluid, where K_f is positive semidefinite and M_f positive definite,
- $C \in \mathbb{R}^{s \times f}$ is due to the coupling effects between fluid and solid,
- $x_s \in \mathbb{R}^s$ is the solid displacement vector, and
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- $x_s \in \mathbb{R}^s$ is the solid displacement vector, and
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The computational solution of Problem (1) with standard sparse eigensolvers involves considerable complications. As a remedy some authors prefer an alternative modeling which additionally involves the fluid displacement potential. Then the resulting system is symmetric and efficient methods such as the shift-and-invert Lanczos method applies. As a drawback, however, the dimension of the problem increases considerably.

Common numerical approach

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One first determines the eigenpairs of the symmetric and definite eigenvalue problems

$$K_S X_S = \omega_S M_S X_S \quad \text{and} \quad K_f X_f = \omega_f M_f X_f \quad (2)$$

by the Lanczos method or automated multi-level sub-structuring, and then projects problem (1) to $\text{diag}\{X_S, X_f\}$, where the columns of X_S and X_f are the eigenmodes of problem (2) not exceeding a given cut-off level.

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The projected problem

$$\begin{bmatrix} X_S^T K_S X_S & X_S^T C X_f \\ 0 & X_f^T K_f X_f \end{bmatrix} \begin{bmatrix} y_S \\ y_f \end{bmatrix} = \lambda \begin{bmatrix} X_S^T M_S X_S & 0 \\ -X_f^T C^T X_S & X_f^T M_f X_f \end{bmatrix} \begin{bmatrix} y_S \\ y_f \end{bmatrix} \quad (3)$$

has the same structure as the original problem but is of much smaller dimension.

Example

Consider a two dimensional coupled structure consisting of steel and air portions discretized by finite elements. The resulting problem has 120473 degrees of freedom, 67616 of which are located in the solid region and 52857 in the fluid part.

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Table: STEEL-AIR-STRUCTURE

coupled	steel	air	rel.dev. [%]	proj. [%]
0.00		0.00		
41.25		41.32	0.16	2.5e-4
48.67		48.71	0.08	2.7e-4
56.96	56.90		0.11	2.2e-3
75.55	75.51		0.06	3.3e-3
93.18		93.19	0.01	1.0e-4
129.99		130.04	0.05	6.1e-4
150.94	151.03		0.06	3.5e-3
158.16		158.18	0.01	1.8e-4
186.64	186.66		0.12	4.2e-3

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If the fluid air is replaced by water, the scene changes completely.

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Table: STEEL-WATER-STRUCTURE

coupled	steel	water	proj.	rel.error [%]
0.00	56.90	0.00	0.00	
28.01	75.51	178.63	28.33	1.2
41.54	151.03	210.64	43.01	3.5
92.73	186.66	402.93	101.98	10.0
124.70	225.54		133.60	7.1
138.26	451.76		141.87	2.6
270.40	472.45		285.18	5.5
321.79			343.80	6.8
388.73			416.87	7.2
402.77			439.83	9.2

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Properties

- (1) can be symmetrized by $T := \begin{pmatrix} M_s^{-1} K_s & M_s^{-1} C \\ 0 & I \end{pmatrix}$, i.e.

$$T^T K X = \begin{pmatrix} K_s M_s^{-1} K_s & K_s M_s^{-1} C \\ C^T M_s^{-1} K_s & K_f + C^T M_s^{-1} C \end{pmatrix} X = \lambda \begin{pmatrix} K_s & 0 \\ 0 & M_f \end{pmatrix} X = \lambda T^T M X$$

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- (1) has only real and non-negative eigenvalues:

$$(x_s^H, x_f^H) T^T K \begin{pmatrix} x_s \\ x_f \end{pmatrix} = (K_s x_s + C x_f)^H M_s^{-1} (K_s x_s + C x_f) + x_f^H K_f x_f \geq 0$$

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- If $x := \begin{pmatrix} x_s \\ x_f \end{pmatrix}$ is a right eigenvector of (1) corresponding to the eigenvalue λ , then $\hat{x} := \begin{pmatrix} \lambda x_s \\ x_f \end{pmatrix}$ is a left eigenvector.

Properties cont.

- Right eigenvectors can be chosen orthonormal with respect to

$$\tilde{M} := \begin{pmatrix} K_s & 0 \\ 0 & M_f \end{pmatrix},$$

left eigenvectors can be chosen orthogonal with respect to

$$\bar{M} := \begin{pmatrix} M_s & 0 \\ 0 & K_f \end{pmatrix}.$$

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- Right eigenvectors x and left eigenvectors \hat{x} corresponding to distinct eigenvalues satisfy

$$\hat{x}^T K x = \hat{x}^T M x = 0.$$

An inverse-free Rayleigh functional

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For a given right eigenvector $\begin{pmatrix} x_s \\ x_f \end{pmatrix}$ corresponding to the eigenvalue λ it holds

$$\lambda = \frac{\begin{pmatrix} \lambda x_s \\ x_f \end{pmatrix}^T \begin{pmatrix} K_s & C \\ 0 & K_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix}}{\begin{pmatrix} \lambda x_s \\ x_f \end{pmatrix}^T \begin{pmatrix} M_s & 0 \\ -C^T & M_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix}} = \frac{\lambda x_s^T K_s x_s + \lambda x_s^T C x_f + x_f^T K_f x_f}{\lambda x_s^T M_s x_s + x_f^T M_f x_f - x_f^T C^T x_s}.$$

An inverse-free Rayleigh functional cont.

This suggests to define a Rayleigh functional p for some general $s + f$ -dimensional vector by the requirement

$$p(x_s, x_f) = \frac{p(x_s, x_f)x_s^T K_s x_s + p(x_s, x_f)x_s^T C x_f + x_f^T K_f x_f}{p(x_s, x_f)x_s^T M_s x_s - x_f^T C^T x_s + x_f^T M_f x_f}$$

which leads to the quadratic equation

$$p(x_s, x_f)^2 x_s^T M_s x_s + p(x_s, x_f)(x_f^T M_f x_f - x_s^T K_s x_s - 2x_s^T C x_f) - x_f^T K_f x_f = 0.$$

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We therefore choose the unique positive root of this equation as Rayleigh functional.

Rayleigh functional

Definition

$$p(x_s, x_f) := \begin{cases} q(x_s, x_f) + \sqrt{q(x_s, x_f)^2 + \frac{x_f^T K_f x_f}{x_s^T M_s x_s}} & \text{if } x_s \neq 0 \\ \frac{x_f^T K_f x_f}{x_f^T M_f x_f} & \text{if } x_s = 0 \end{cases} \quad (4)$$

where

$$q(x_s, x_f) := \frac{x_s^T K_s x_s - x_f^T M_f x_f + 2x_s^T C x_f}{2x_s^T M_s x_s}$$

is called Rayleigh functional of the fluid-solid vibration eigenvalue problem (1).

Properties

Well-known properties for symmetric eigenproblems can be generalized to the fluid-solid eigenproblem using the Rayleigh functional (4)

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Differentiating the defining equation

$$\rho(x_s, x_f)^2 x_s^T M_s x_s + \rho(x_s, x_f)(x_f^T M_f x_f - x_s^T K_s x_s - 2x_s^T C x_f) - x_f^T K_f x_f = 0$$

of the Rayleigh functional yields

Lemma

Any right eigenvector x of problem (1) is a stationary point of p , i.e.

$$\nabla p(x) = 0.$$

Properties, cont.

Lemma

Assume that

$$x = \sum_{i \in I} \alpha_i x_i$$

is a linear combination of some eigenvectors indexed by I .

Then the Rayleigh functional is bounded by

$$\min_{i \in I} \lambda_i \leq \rho(x) \leq \max_{i \in I} \lambda_i.$$

Variational characterizations

As for symmetric eigenproblems one can derive variational characterizations of eigenvalues.

Let $\lambda_1 \leq \lambda_2 \leq \dots \lambda_{s+f}$ be the eigenvalues of problems (1) and $\tilde{M} := \text{diag}\{K_s, M_f\}$. Then it holds that

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Proposition (Rayleigh's principle)

$$\begin{aligned}\lambda_k &= \min\{\rho(x) : x^T \tilde{M} x_j = 0, j = 1, \dots, k-1\} \\ &= \max\{\rho(x) : x^T \tilde{M} x_j = 0, j = k+1, \dots, s+f\}\end{aligned}$$

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Proposition (Rayleigh's principle)

$$\begin{aligned}\lambda_k &= \min\{p(x) : x^T \tilde{M} x_j = 0, j = 1, \dots, k-1\} \\ &= \max\{p(x) : x^T \tilde{M} x_j = 0, j = k+1, \dots, s+f\}\end{aligned}$$

Proposition (Minmax characterization)

$$\lambda_k = \min_{\dim S_k = k} \max_{0 \neq x \in S_k} p(x) = \max_{\dim S_k = s+f+1-k} \min_{0 \neq x \in S_k} p(x).$$

Structure preserving projection

Proposition

Assume that $V = \begin{pmatrix} V_s & 0 \\ 0 & V_f \end{pmatrix} \in \mathbb{R}^{s+f \times k}$ has maximal rank k .

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and let $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots \leq \tilde{\lambda}_k$ be the eigenvalues of the projected problem

$$K_V y = \tilde{\lambda} M_V y.$$

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and let $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots \leq \tilde{\lambda}_k$ be the eigenvalues of the projected problem

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Then it holds that

$$\lambda_j \leq \tilde{\lambda}_j, \quad j = 1, \dots, k.$$

In particular, the j th eigenvalue λ_j does not exceed the j th eigenvalue of the structure eigenproblems $K_s x_s = \lambda M_s x_s$ and of the fluid eigenvalue problem $K_f x_f = \lambda M_f x_f$.

Rayleigh functional iteration

Require: initial vector $x^{(0)}$, $k = 0$

1: **repeat**

2: evaluate Rayleigh functional $\rho_k := \rho(x^{(k)})$

3: solve $(K - \rho_k M)x^{(k+1)} = Mx^{(k)}$ for $x^{(k+1)}$

4: Normalize $x^{(k+1)}$ by \tilde{M}

5: $k \leftarrow k + 1$

6: **until** convergence

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Similar as for the Rayleigh quotient iteration for symmetric eigenproblems, local cubic convergence can be shown.

Proposition

The iterates ρ_k and $x^{(k)}$ converge locally cubically towards an eigenvalue λ and a corresponding eigenvector x .

Iterative projection methods

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To overcome these drawbacks, one considers iterative projection methods where the projection space V is expanded by an approximation to the direction of the Rayleigh functional iteration.

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Iterative projection method

Require: Initial basis V with $V^H V = I$; set $m = 1$

- 1: **while** $m \leq$ number of wanted eigenvalues **do**
- 2: compute eigenpair (μ, y) of projected problem $V^T(K - \mu M)Vy = 0$.
- 3: determine Ritz vector $u = Vy$, $\|u\| = 1$, and residual $r = (K - \mu M)u$
- 4: **if** $\|r\| < \varepsilon$ **then**
- 5: accept approximate eigenpair $\lambda_m = \mu$, $x_m = u$; increase $m \leftarrow m + 1$
- 6: reduce search space V if necessary
- 7: choose approximation (λ_m, u) to next eigenpair, and compute $r = (K - \lambda_m M)u$
- 8: **end if**
- 9: expand search space $V = [V, v_{\text{new}}]$
- 10: **end while**

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Main tasks

- expand search space
- choose eigenpair of projected problem (locking, purging)

Jacobi-Davidson method

Expanding the current subspace V_k by $x^{(k+1)} := (K - \rho_k M)^{-1} M x^{(k)}$ is equivalent to expanding it by $t := x^{(k+1)} + \alpha x^{(k)}$ for every $\alpha \in \mathbb{R}$.

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The most robust expansion of this type with respect to inexact solves of $(K - \rho_k M)x^{(k+1)} = Mx^{(k)}$ satisfies $x^{(k)T} \tilde{M}t = 0$ (cf. V. 2007)

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which leads to the equivalent so called correction equation $(\hat{M} := \text{diag}\{M_s, K_f\})$

$$\left(I - \frac{\hat{M} x x^T}{x^T \hat{M} x}\right) (K - \rho_k M) \left(I - \frac{x x^T \tilde{M}}{x^T \tilde{M} x}\right) t = -(K - \rho_k M) x, \quad t^T \tilde{M} x = 0$$

for a given eigenpair approximation (ρ_k, x) .

Jacobi-Davidson method

Expanding the current subspace V_k by $x^{(k+1)} := (K - \rho_k M)^{-1} M x^{(k)}$ is equivalent to expanding it by $t := x^{(k+1)} + \alpha x^{(k)}$ for every $\alpha \in \mathbb{R}$.

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Iterative projection methods of this type are known as Jacobi-Davidson method and were introduced by Sleijpen and van der Vorst (1996).

structure preserving Jacobi-Davidson method

The standard approach of the Jacobi-Davidson method, i.e. considering

$$V^T K V y = \lambda V^T M V y \quad (6)$$

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To preserve the structure of the fluid-solid eigenproblems and to ensure real eigenvalues of the projected eigenproblem we expand the ansatz space in every step by 2 vectors $\begin{pmatrix} t_s \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ t_f \end{pmatrix}$ where $t = \begin{pmatrix} t_s \\ t_f \end{pmatrix}$ solves approximately the correction equation. Hence, we use structure preserving projection matrices

$$V = \begin{pmatrix} V_s & 0 \\ 0 & V_f \end{pmatrix}$$

and obtain

$$V^T K V = \begin{pmatrix} V_s^T K_s V_s & V_s^T C V_f \\ 0 & V_f^T K_f V_f \end{pmatrix} \quad \text{and} \quad V^T M V = \begin{pmatrix} V_s^T M_s V_s & 0 \\ -V_f^T C^T V_s & V_f^T M_f V_f \end{pmatrix}.$$

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From the minmax characterization we obtain the following monotonicity result:

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Proposition

Let $\lambda_j^{(k)}$ be the j smallest eigenvalue of the k th projected eigenproblem of the structure preserving Jacobi-Davidson method.

Then it holds that

$$\lambda_j \leq \lambda_j^{(k+1)} \leq \lambda_j^{(k)} \text{ for } k = 1, \dots, \dim(V_k).$$

Structure preserving Jacobi-Davidson method

REQUIRE Initial basis $V = \text{diag}\{V_s, V_f\}$, $V_s^T K_s V_s = I$, $V_f^T M_f V_f = I$,
 $m = 1$; $\theta_m = 0$;

1: determine preconditioner $L \approx (K - \sigma M)^{-1}$, $\sigma \approx \lambda_{\min}$

2: **while** $\theta_m \leq \text{maxeig}$ **do**

3: solve the projected problem

$$\begin{pmatrix} V_s^T K_s V_s & V_s^T C V_f \\ 0 & V_f^T K_f V_f \end{pmatrix} \begin{pmatrix} y_s \\ y_f \end{pmatrix} = \theta \begin{pmatrix} V_s^T M_s V_s & 0 \\ -V_f^T C^T V_s & V_f^T M_f V_f \end{pmatrix} \begin{pmatrix} y_s \\ y_f \end{pmatrix}$$

4: choose m smallest eigval. θ_m and corresp. eigvec. $(y_s^T, y_f^T)^T$

5: determine Ritz vector $x = \begin{pmatrix} V_s y_s \\ V_f y_f \end{pmatrix}$ and residual $r = (K - \theta_m M)x$

6: **if** $\|r\|/\|x\| < \epsilon$

7: **while** $\|r\|/\|x\| < \epsilon$

8: accept approximate m th eigenpair (θ_m, x) ; increase $m \leftarrow m + 1$;

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10: determ. Ritz vec. $x = \begin{pmatrix} V_s y_s \\ V_f y_f \end{pmatrix}$ and res. $r = (K - \theta_m M)x$

11: **endwhile**

- 12: reduce search space V if indicated
 13: determine new preconditioner $L \approx (K - \theta M)^{-1}$ if necessary
 14: **endif**
 15: compute approximate solution $t = (t_s^T, t_f^T)^T$ of the correction equation

$$\left(I - \frac{\hat{M}xx^T}{x^T \hat{M}x}\right)(K - \rho_m M)\left(I - \frac{xx^T \tilde{M}}{x^T \tilde{M}x}\right)t = r, x^T \tilde{M}t = 0$$

- 16: orthogonalize $v_s = t_s - K_s V_s V_s^T K_s t_s$, $v_f = t_f - M_f V_f V_f^T M_f t_f$
 17: **if** $\|v_s\|_{K_s} > \text{tol}$ expand $V_s \leftarrow [V_s, v_s / \|v_s\|_{K_s}]$
 18: **if** $\|v_f\|_{M_f} > \text{tol}$ expand $V_f \leftarrow [V_f, v_f / \|v_f\|_{M_f}]$
 19: update projected problem
 20: **endwhile**

Solving correction equation

The correction equation is solved approximately by a few steps of an iterative solver (GMRES or BiCGStab).

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The operator $T(\rho) := K - \rho M$ is restricted to map the subspace x^\perp into itself. Hence, if $L \approx T(\sigma)^{-1}$ is a preconditioner of $T(\sigma)$, then a preconditioner for an iterative solver of the correction equation should be modified correspondingly to

$$\tilde{L} := \left(I - \frac{\hat{M} x x^T}{x^T \hat{M} x} \right) L \left(I - \frac{x x^T \tilde{M}}{x^T \tilde{M} x} \right).$$

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Taking into account the projectors in the preconditioner, i.e. using \tilde{L} instead of L , raises the cost of the preconditioned Krylov solver only slightly (cf. Sleijpen, van der Vorst). Only one additional linear solve with preconditioner L is required.

Nonlinear Arnoldi method

Alternatively one may expand the search space by the Cayley transformation

$$t_{Ct} = (K - \sigma M)^{-1}(K - \theta M)x$$

where σ is a parameter close to the wanted eigenvalue (which is kept fixed for several iteration steps), and (θ, x) is the current approximation to the eigenpair wanted next.

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Notice that it holds that

$$(\theta - \sigma)t_{ji} = (\theta - \sigma)(K - \sigma M)^{-1}Mx = x - (K - \sigma M)^{-1}(K - \theta M)x,$$

and since the current Ritz vector x is already contained in V the expansions t_{Ct} and inverse iteration

$$t_{ji} = (K - \sigma M)^{-1}Mx$$

are equivalent.

Arnoldi type iterative projection method

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1: determine preconditioner $L \approx (K - \sigma M)^{-1}$, $\sigma \approx \lambda_{\min}$

2: **while** $\theta_m \leq \text{maxeig}$

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$$\begin{pmatrix} V_s^T K_s V_s & V_s^T C V_f \\ 0 & V_f^T K_f V_f \end{pmatrix} \begin{pmatrix} y_s \\ y_f \end{pmatrix} = \theta \begin{pmatrix} V_s^T M_s V_s & 0 \\ -V_f^T C^T V_s & V_f^T M_f V_f \end{pmatrix} \begin{pmatrix} y_s \\ y_f \end{pmatrix}$$

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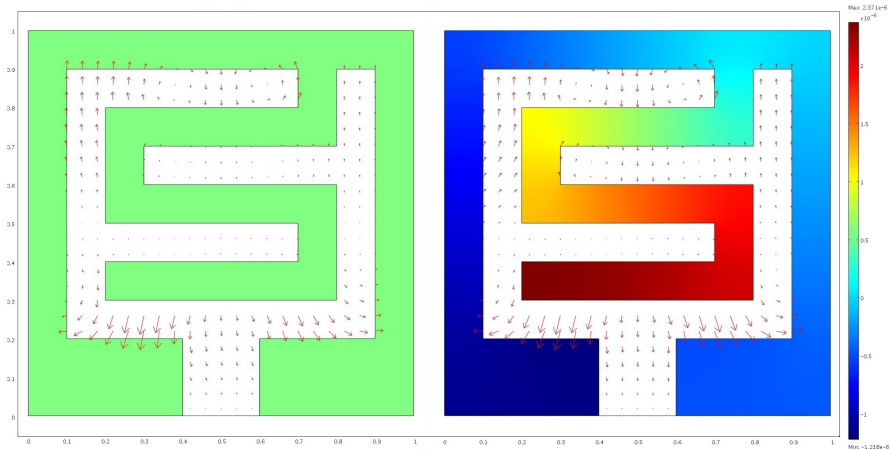
11: **endwhile**

- 12: reduce search space V if indicated
- 13: determine new preconditioner $L \approx (K - \theta M)^{-1}$ if necessary
- 14: **endif**
- 15: solve $Lt = r$ for $t = (t_s^T, t_f^T)^T$
- 16: orthogonalise $v_s = t_s - K_s V_s V_s^T t_s$, $v_f = t_f - M_f V_f V_f^T t_f$
- 17: **if** $\|v_m\| > \text{tol}$ expand $V_m \leftarrow [V_m, v_m/\|v_m\|]$ **endif**
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- 20: **endwhile**

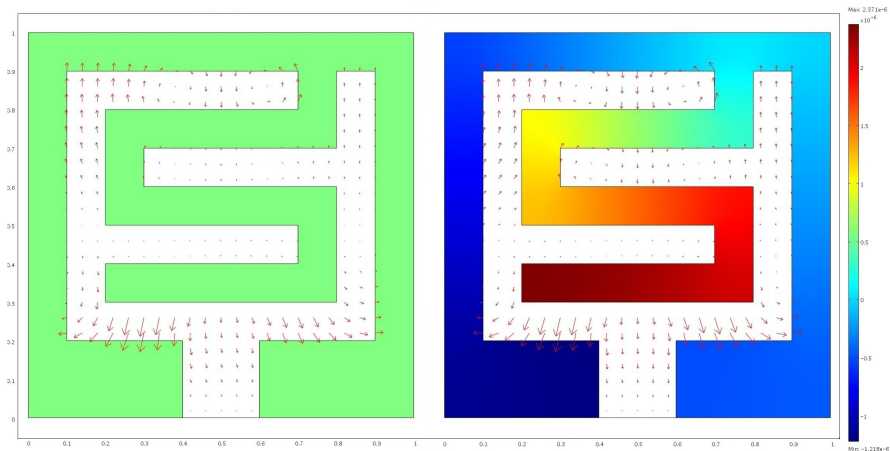
Outline

- 1 Problem definition
- 2 An (inverse-free) Rayleigh functional
- 3 Structure preserving iterative projection methods
- 4 Numerical Results**
- 5 AMLS

Numerical Example

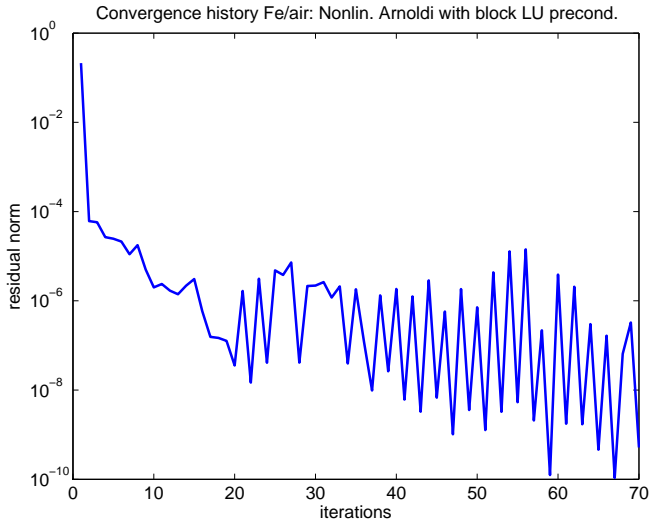


Numerical Example



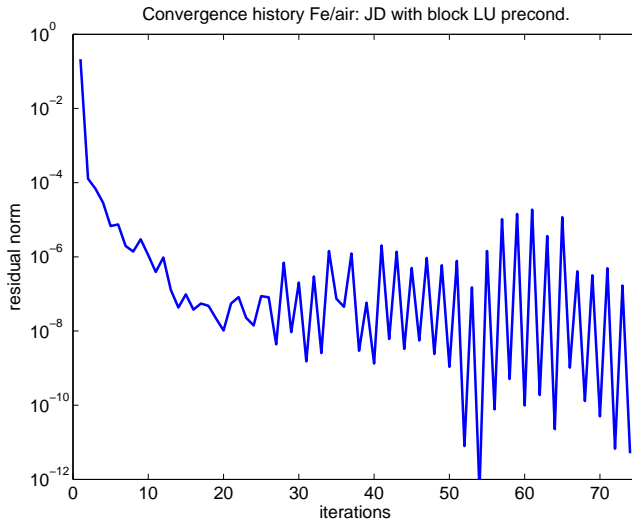
Consider a finite element model of a coupled steel/air system with 120473 DoFs (67616 AI, 52857 air).
 Determine all 23 eigenfrequencies less than 500 Hz.

Numerical Example; Fe-air



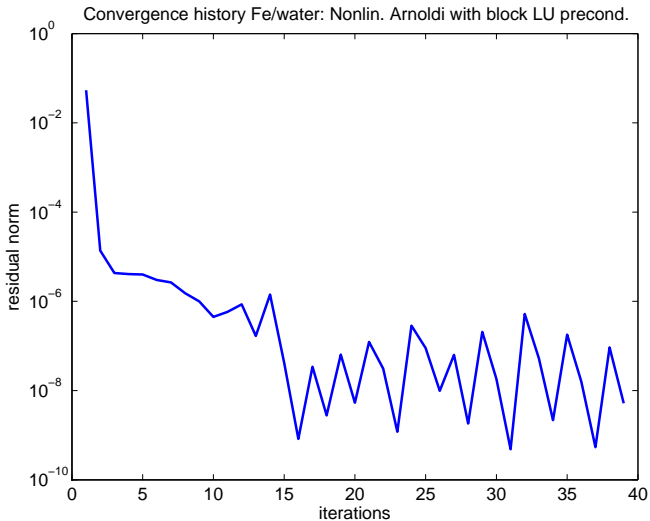
CPU time (Pentium D, 3.4 GHz, 4GB) : 19.5 sec.

Numerical Example; Fe-air



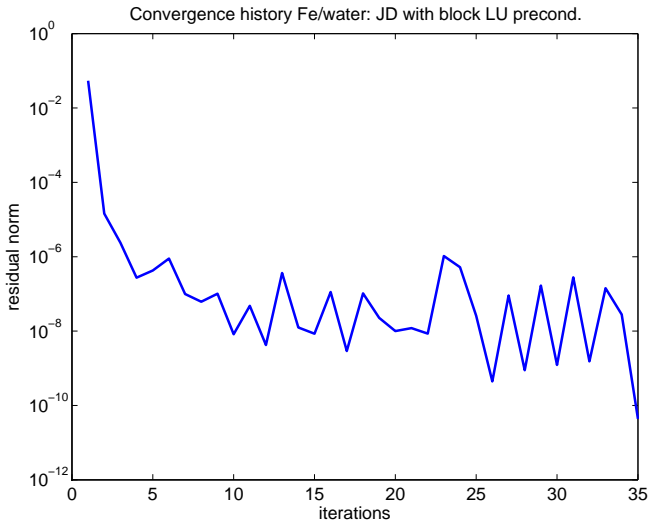
CPU time : 77.9 sec.

Numerical Example; Fe-H₂O



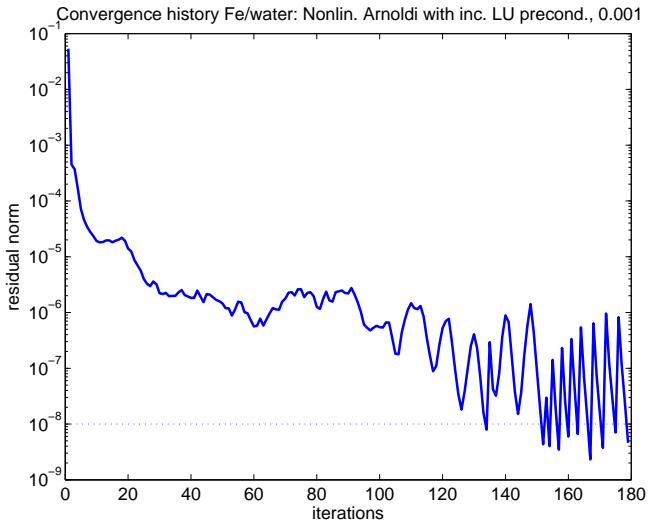
CPU time : 10.4 sec.

Numerical Example; Fe-H₂O



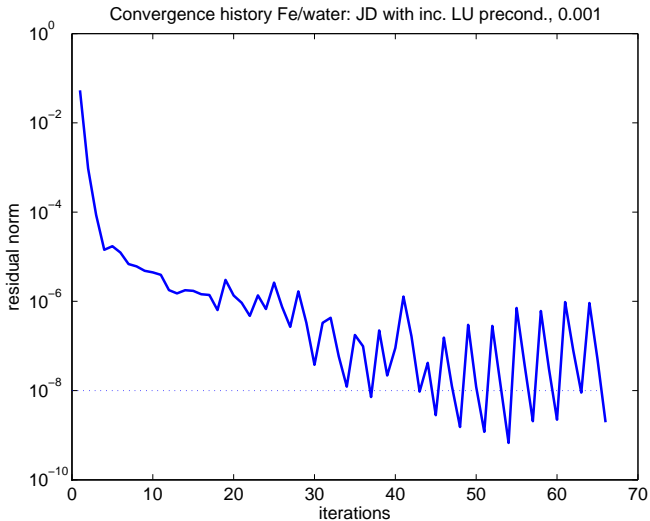
CPU time : 30.0 sec.

Numerical Example; Fe-H₂O



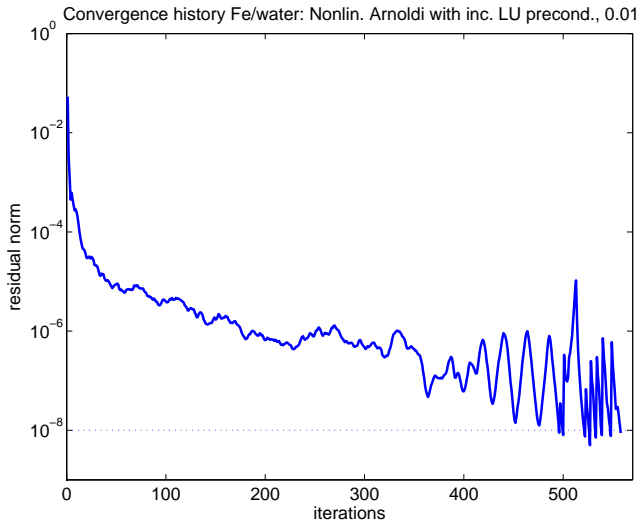
CPU time : 113.4 sec.

Numerical Example; Fe-H₂O



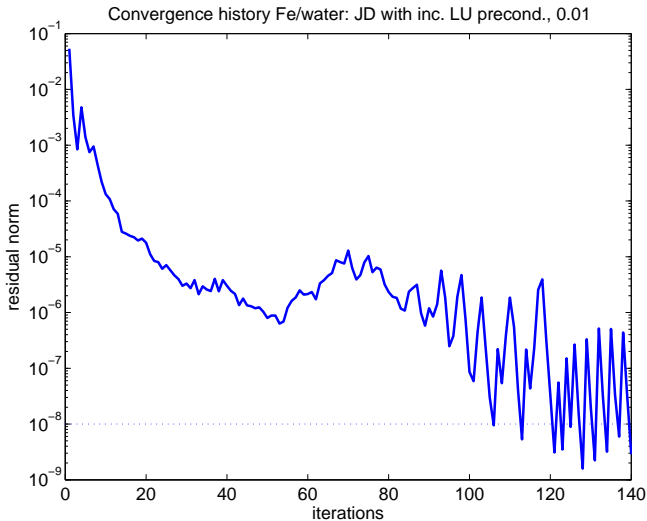
CPU time : 60.5 sec.

Numerical Example; Fe-H₂O



CPU time : 3411.1 sec.

Numerical Example; Fe-H₂O



CPU time : 125.6 sec.

Conclusions

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- Rayleigh functional iteration converges locally and cubically to eigenvalues and eigenvectors.
- Structure preserving iterative projection methods like Nonlinear Arnoldi or Jacobi–Davidson converge fast and robust.
- For accurate preconditioners the Nonlinear Arnoldi method is superior to the Jacobi–Davidson method, but JD is much more robust with respect to coarse preconditioners.

Outline

- 1 Problem definition
- 2 An (inverse-free) Rayleigh functional
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- 5 **AMLS**

Automated Multi-Level Substructuring

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Assuming that the interior degrees of freedom of substructures depend quasistatically on the interface degrees of freedom, and modeling the deviation from quasistatic dependence in terms of a small number of selected substructure eigenmodes the size of the finite element model is reduced substantially yet yielding satisfactory accuracy over a wide frequency range of interest.

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Recent studies in vibro-acoustic analysis of passenger car bodies where very large FE models with more than six million degrees of freedom appear and several hundreds of eigenfrequencies and eigenmodes are needed have shown that AMLS is considerably faster than Lanczos type approaches.

Usual approach in automotive industry

Solve the uncoupled eigenvalue problem

$$K_U X := \begin{pmatrix} K_s & 0 \\ 0 & K_f \end{pmatrix} \begin{pmatrix} v_s \\ v_f \end{pmatrix} = \lambda \begin{pmatrix} M_s & 0 \\ 0 & M_f \end{pmatrix} \begin{pmatrix} v_s \\ v_f \end{pmatrix} =: \lambda M_U X,$$

and project the original problem to $\text{diag}\{\Phi_s, \Phi_f\}$, where the columns of Φ_s and Φ_f are eigenmodes corresponding to eigenvalues not exceeding a cut-off frequency.

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For strongly coupled problems some eigenpairs are inexact.

Example: Conca et al. 1995

Determine acoustic eigenfrequencies of a (very long) cavity containing a tube bundle.

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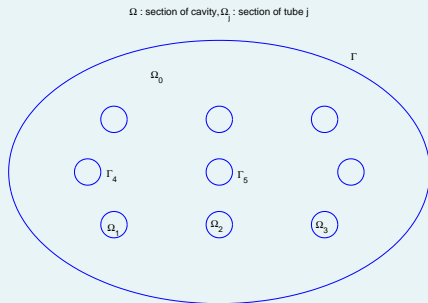
- immersed in an inviscid compressible fluid
- rigid, assembled in parallel inside the fluid,
- elastically mounted such that they can vibrate transversally, but can not move in the direction perpendicular to their sections.

Example: Conca et al. 1995

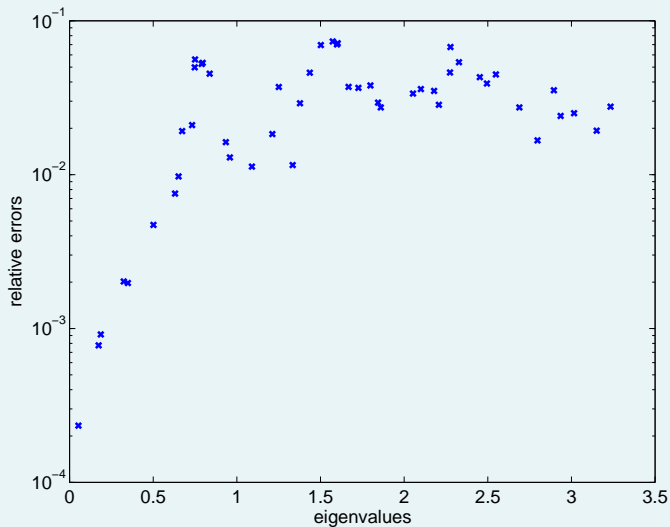
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- rigid, assembled in parallel inside the fluid,
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Due to these assumptions, three-dimensional effects are neglected, and the problem is studied in any transversal section of the cavity.

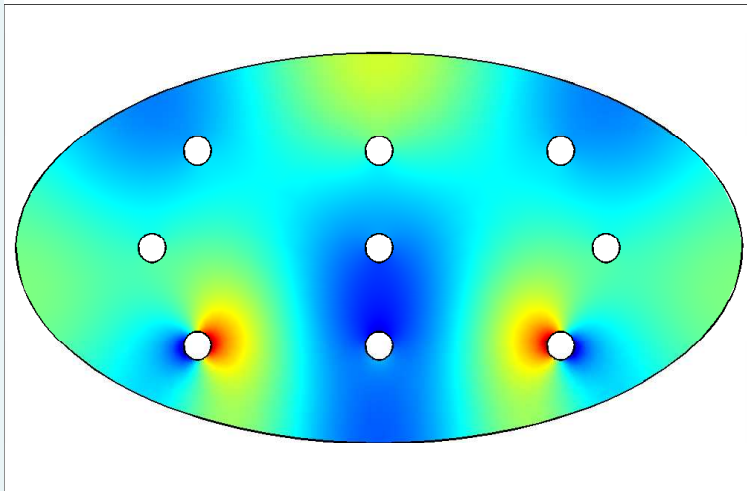


Finite element model: 143081 DoFs



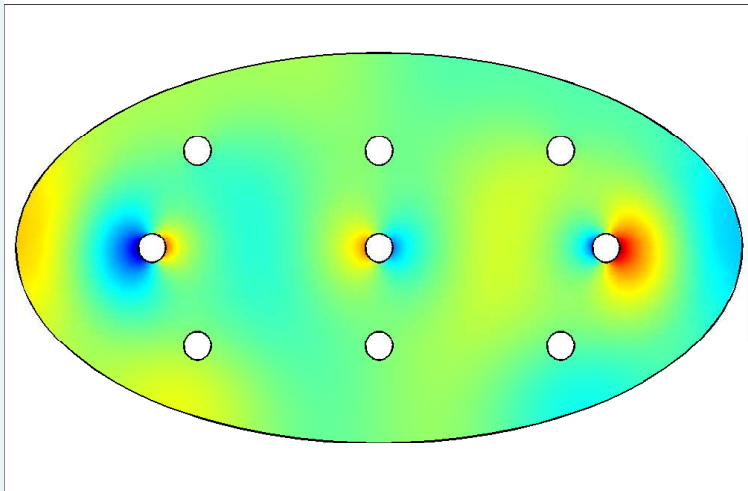
Typical eigenvector

$\lambda(13)=0.7506$



Typical eigenvector

$\lambda(27)=1.5988$



Structure preserving AMLS

Substructure the joint graph of

$$K := \begin{pmatrix} K_s & C \\ 0 & K_f \end{pmatrix} \quad \text{and} \quad M := \begin{pmatrix} M_s & 0 \\ -C^T & M_f \end{pmatrix}$$

substructured recursively on several levels, where the coupling matrix is incorporated into the substructuring process. Hence, any substructure may consist solely of degrees of freedom from the fluid or from the solid, or caused by the coupling matrix it may contain degrees of freedom of either type.

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The crucial point is to modify the AMLS algorithm such that the structure is preserved for the reduced problem, and all eigenvalues are still real.

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Pure fluid or solid substructures obviously can be treated in the same way as in the symmetric AMLS method, i.e. they can be decoupled in the stiffness matrix by Gaussian elimination and reduced by modal truncation. We now describe a typical general reduction step.

Structure preserving AMLS

After a couple of reduction steps one arrives at the following matrices where the unknowns have been reordered appropriately

$$\begin{pmatrix} K_{pp} & 0 & K_{pl}^f & 0 & K_{pi}^f \\ K_{lp}^s & K_{ll}^s & C_l & K_{li}^s & C_{li} \\ 0 & 0 & K_{ll}^f & 0 & K_{li}^f \\ K_{ip}^s & K_{il}^s & C_{il} & K_{ii}^s & C_i \\ 0 & 0 & K_{il}^f & 0 & K_{ii}^f \end{pmatrix}, \begin{pmatrix} K_{pp} & M_{pl}^s & M_{pl}^f & M_{pi}^s & M_{pi}^f \\ M_{lp}^s & M_{ll}^s & 0 & M_{li}^s & 0 \\ M_{lp}^f & -C_l^T & M_{ll}^f & -C_{il}^T & M_{li}^f \\ M_{ip}^s & M_{il}^s & 0 & M_{ii}^s & 0 \\ M_{ip}^f & -C_{li}^T & M_{il}^f & -C_i^T & M_{ii}^f \end{pmatrix}. \quad (7)$$

Here p denotes the degrees of freedom obtained in the reduction steps on previous levels, l collects the degrees of freedom to be handled in the current step, i corresponds to the index set of parent substructures.

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Next one annihilates K_{li}^s and K_{li}^f by symmetric block Gaussian elimination, and ...

Structure preserving AMLS

solving the substructure eigenvalue problem

$$\mathbf{K}_\ell \begin{pmatrix} \phi \\ \psi \end{pmatrix} := \begin{pmatrix} \mathbf{K}_{\ell\ell}^s & \mathbf{C}_\ell \\ \mathbf{0} & \mathbf{K}_{\ell\ell}^f \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \omega \begin{pmatrix} \mathbf{M}_{\ell\ell}^s & \mathbf{0} \\ -\mathbf{C}_\ell^T & \mathbf{M}_{\ell\ell}^f \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} =: \mathbf{M}_\ell \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

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$$\mathbf{K}_\ell \begin{pmatrix} \phi \\ \psi \end{pmatrix} := \begin{pmatrix} \mathbf{K}_{\ell\ell}^s & \mathbf{C}_\ell \\ \mathbf{0} & \mathbf{K}_{\ell\ell}^f \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \omega \begin{pmatrix} \mathbf{M}_{\ell\ell}^s & \mathbf{0} \\ -\mathbf{C}_\ell^T & \mathbf{M}_{\ell\ell}^f \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} =: \mathbf{M}_\ell \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

and normalizing the eigenvectors such that

$$\left(\phi_i^T, \frac{1}{\omega_i} \psi_i^T \right) \begin{pmatrix} \mathbf{M}_{\ell\ell}^s & \mathbf{0} \\ -\mathbf{C}_\ell^T & \mathbf{M}_{\ell\ell}^f \end{pmatrix} \begin{pmatrix} \phi_j \\ \psi_j \end{pmatrix} = \delta_{ij}$$

Structure preserving AMLS

solving the substructure eigenvalue problem

$$\mathbf{K}_\ell \begin{pmatrix} \phi \\ \psi \end{pmatrix} := \begin{pmatrix} \mathbf{K}_{\ell\ell}^s & \mathbf{C}_\ell \\ \mathbf{0} & \mathbf{K}_{\ell\ell}^f \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \omega \begin{pmatrix} \mathbf{M}_{\ell\ell}^s & \mathbf{0} \\ -\mathbf{C}_\ell^T & \mathbf{M}_{\ell\ell}^f \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} =: \mathbf{M}_\ell \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

and normalizing the eigenvectors such that

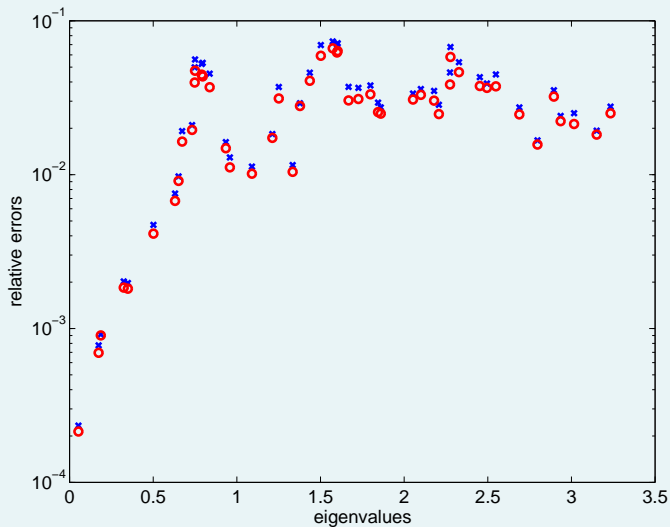
$$\begin{pmatrix} \phi_i^T, \frac{1}{\omega_i} \psi_i^T \end{pmatrix} \begin{pmatrix} \mathbf{M}_{\ell\ell}^s & \mathbf{0} \\ -\mathbf{C}_\ell^T & \mathbf{M}_{\ell\ell}^f \end{pmatrix} \begin{pmatrix} \phi_j \\ \psi_j \end{pmatrix} = \delta_{ij}$$

the oblique projection

$$\begin{pmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Phi_\ell^T & \Omega^{-1} \Psi_\ell^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I \end{pmatrix} (\mathbf{K}_\ell - \lambda \mathbf{M}_\ell) \begin{pmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Phi_\ell & \mathbf{0} \\ \mathbf{0} & \Psi_\ell & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{pmatrix} = \mathbf{0}$$

preserves the structure of (7).

Structure preserving AMLS



AMLS for fluid-solid vibrations

Consider the symmetric eigenproblem

$$\left[\begin{pmatrix} 0 & C & K_s & 0 \\ C^T & 0 & 0 & K_f \\ K_s & 0 & 0 & 0 \\ 0 & K_f & 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} M_s & 0 & 0 & 0 \\ 0 & M_f & 0 & 0 \\ 0 & 0 & K_s & 0 \\ 0 & 0 & 0 & K_f \end{pmatrix} \right] x = 0$$

AMLS for fluid-solid vibrations

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If $(\lambda^2, (x_s^T, x_f^T)^T)$ is an eigenpair of the fluid-solid-problem then

$$(\pm\lambda, (\lambda^2 x_s^T \quad \pm\lambda x_f^T \quad \pm\lambda x_s^T \quad x_f^T)^T)$$

are solutions of the symmetric problem above unless $\lambda = 0$.

AMLS for fluid-solid vibrations

Consider the symmetric eigenproblem

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are solutions of the symmetric problem above unless $\lambda = 0$.

Partitioning based on the union of the sparsity structures of the matrices K and M gives an $(s + f)$ -dimensional partitioning which can be expanded to an $2(s + f)$ -dimensional partitioning so that for $i = 1, \dots, s + f$ the i th and $(i + s + f)$ th degree of freedom belong to the same substructure or interface.

AMLS for fluid-solid vibrations

Consider the symmetric eigenproblem

$$\left[\begin{pmatrix} 0 & C & K_s & 0 \\ C^T & 0 & 0 & K_f \\ K_s & 0 & 0 & 0 \\ 0 & K_f & 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} M_s & 0 & 0 & 0 \\ 0 & M_f & 0 & 0 \\ 0 & 0 & K_s & 0 \\ 0 & 0 & 0 & K_f \end{pmatrix} \right] x = 0$$

If $(\lambda^2, (x_s^T, x_f^T)^T)$ is an eigenpair of the fluid-solid-problem then

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Partitioning based on the union of the sparsity structures of the matrices K and M gives an $(s + f)$ -dimensional partitioning which can be expanded to an $2(s + f)$ -dimensional partitioning so that for $i = 1, \dots, s + f$ the i th and $(i + s + f)$ th degree of freedom belong to the same substructure or interface.

Standard AMLS can be performed at nearly the same cost as AMLS for an $(s + f)$ dimensional problem.

AMLS for fluid-solid vibrations

