A Jacobi-Davidson method for two real parameter nonlinear eigenvalue problems arising from delay differential equations

Heinrich Voss
voss@tuhh.de

Joint work with Karl Meerbergen (KU Leuven) and Christian Schröder (TU Berlin)

TU Hamburg–Harburg

TUHH
Technische Universität Hamburg-Harburg

Outline

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Problem

Consider the linear time-invariant delay differential equation

$$\boxed{M\dot{x}(t) + Ax(t) + Bx(t-\tau) = 0}$$

with given $M, A, B \in \mathbb{C}^{n \times n}$; $\tau \geq 0$ is the delay

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Necessary and sufficient is that the spectrum of the nonlinear eigenvalue problem

$$\boxed{\lambda Mu + Au + e^{-\tau\lambda}Bu = 0}$$

is contained in the open left half–plane.

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Wanted: *critical delays* τ where system changes stability
necessary: system has a purely imaginary eigenvalue

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Solving small critical delay problems

Recall the DEVP

$$(i\omega M + A + e^{-i\omega\tau} B)u = 0$$

where $M, A, B \in \mathbb{C}^{n\times n}$.

We are interested in solutions $(\omega, \tau, u) \in \mathbb{R} \times \mathbb{R} \times \mathbb{C}^n$, $u \neq 0$.

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Hence the complex conjugate equation reads

$$-i\omega\bar{M}\bar{u} + \bar{A}\bar{u} + \mu^{-1}\bar{B}\bar{u} = 0.$$

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$$\begin{aligned} 0 &= -(i\omega Mu) \otimes \bar{M}\bar{u} - Mu \otimes (-i\omega \bar{M}\bar{u}) \\ &= (Au + \mu Bu) \otimes \bar{M}\bar{u} + Mu \otimes (\bar{A}\bar{u} + \mu^{-1}\bar{B}\bar{u}) \\ &= ((A + \mu B) \otimes \bar{M} + M \otimes (\bar{A} + \mu^{-1}\bar{B}))(u \otimes \bar{u}). \end{aligned}$$

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Expanding and multiplication by μ yields the quadratic eigenvalue problem

$$\boxed{\mu^2(B \otimes \bar{M})z + \mu(A \otimes \bar{M} + M \otimes \bar{A})z + (M \otimes \bar{B})z = 0.}$$

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We have thus shown that if (ω, τ, u) is a solution of the DEVP and $\mu = e^{-i\omega\tau}$ is on the unit circle, then $(\mu, u\otimes\bar{u})$ is an eigenpair of the quadratic eigenvalue problem (QEVP) above.

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Let $M, A, B \in \mathbb{C}^{n,n}$ with M nonsingular. Then any eigenvector $z \in \mathbb{C}^{n^2}$ of QEVP corresponding to a simple eigenvalue $\mu \in \mathbb{C}$ can be written as $z = \alpha u_1 \otimes u_2$ for some vectors $u_1, u_2 \in \mathbb{C}^n$ and some $\alpha \in \mathbb{C}$.

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Moreover, if $|\mu| = 1$ then $u_1 = \bar{u}_2 = u$ and there is an $\omega \in \mathbb{R}$ such that DEVP holds.

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We have thus transformed the problem of finding solutions of the DEVP to the problem of finding eigenvalues of modulus one of a QEVP.

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This can be done by the structure preserving method similar to the method presented in Fassbender,Mackey,Mackey,Schröder (2008)

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Finally, τ may be computed from μ and ω as $\tau = -\mathrm{Im}(\ln(\mu))/\omega$.

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Hence, QEVP can be rewritten as

$$\boxed{(\mu^2 A_2 + \mu A_1 + A_0)z = 0}$$

where

$$\boxed{A_2 = P\bar{A}_0 P, \quad A_1 = P\bar{A}_1 P, \quad \text{whith } P = P^{-1} = P^T \in \mathbb{R}^{n^2 \times n^2}.}$$

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Such problems are called PCP-palindromic eigenvalue problems.

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$$\left(\mu \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} \bar{A}_1 - \bar{A}_2 & \bar{A}_0 \\ \bar{A}_0 & \bar{A}_0 \end{bmatrix} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} + \begin{bmatrix} A_1 - A_2 & A_0 \\ A_0 & A_0 \end{bmatrix}\right) \begin{bmatrix} \mu z \\ z \end{bmatrix} = 0,$$

which is a linear PCP-palindromic problem, because with P also $\left[\begin{smallmatrix} & P \\ P & \end{smallmatrix}\right]$ is a real symmetric permutation.

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Second, using the factorization

$$\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} = UU^T, \quad \text{where} \quad U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & iP \\ P & -iI \end{bmatrix}$$

we define

$$C := U^H \begin{bmatrix} A_1 - A_2 & A_0 \\ A_0 & A_0 \end{bmatrix} \bar{U}.$$

Structure preserving method for QEVP

Since U is unitary, i.e. $U^H U = I = U^T \bar{U}$, this pencil is equivalent to

$$
\begin{aligned}
&U^H \left(\mu \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} \bar{A}_1 - \bar{A}_2 & \bar{A}_0 \\ \bar{A}_0 & \bar{A}_0 \end{bmatrix} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} + \begin{bmatrix} A_1 - A_2 & A_0 \\ A_0 & A_0 \end{bmatrix} \right) \bar{U} \\
=&\mu U^T \begin{bmatrix} \bar{A}_1 - \bar{A}_2 & \bar{A}_0 \\ \bar{A}_0 & \bar{A}_0 \end{bmatrix} U + U^H \begin{bmatrix} A_1 - A_2 & A_0 \\ A_0 & A_0 \end{bmatrix} \bar{U} \\
=&\mu \bar{C} + C.
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Third, consider an eigenpair (θ, x) of the real generalized eigenvalue problem

$$\mathrm{Re}(C)x = \theta \mathrm{Im}(C)x$$

or, equivalently,

$$Cx + \frac{i + \theta}{i - \theta} \bar{C}x = 0.$$

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Note that $\mu := (i+\theta)/(i-\theta)$ is on the unit circle if and only if θ is real or $\theta = \infty$ (the latter resulting in $\mu = 1$). Note that real simple eigenvalues of real pencils can be stably computed by e.g. the real QZ algorithm.

Structure preserving method for QEVP

Require: $M, A, B \in \mathbb{C}^{n\times n}$

Ensure: solutions $(\omega_j, \tau_j, u_j)_{j=1,\dots}$ of $(i\omega M + A + e^{-i\omega\tau}B)u = 0$

$A_0 = M \otimes \bar{B}$, $A_1 = A \otimes \bar{M} + M \otimes \bar{A}$

Construct the permutation P

$C = \frac{1}{2}\begin{bmatrix} I & iP \\ P & -iI \end{bmatrix}^H \begin{bmatrix} A_1 - P\overline{A_0}P & A_0 \\ A_0 & A_0 \end{bmatrix} \overline{\begin{bmatrix} I & iP \\ P & -iI \end{bmatrix}}$

Compute all eigenpairs (θ_j, x_j) of $\mathrm{Re}(C)x = \theta\mathrm{Im}(C)x$ where θ_j is real

for j=1,... **do**

  $\mu_j = \frac{i+\theta_j}{i-\theta_j}$

  $z_j = [I \;\; -iP]x_j$

  Compute u_j as dominant singular vector of $\mathrm{mat}(z_j)$

  $\omega_j = \frac{-\mathrm{Im}((Mu_j)^H(Au_j+\mu_j Bu_j))}{\|Mu_j\|_2^2}$

  $\tau_j = \frac{-\mathrm{Im}(\ln(\mu_j))}{\omega_j}$

end for

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5: **for** j=1,... **do**

6: $\quad \mu_j = \frac{i+\theta_j}{i-\theta_j}$

7: $\quad z_j = [I \; -iP]x_j$

8: $\quad$ Compute u_j as dominant singular vector of $\mathrm{mat}(z_j)$

9: $\quad \omega_j = \frac{-\mathrm{Im}((Mu_j)^H(Au_j + \mu_j Bu_j))}{\|Mu_j\|_2^2}$

10: $\quad \tau_j = \frac{-\mathrm{Im}(\ln(\mu_j))}{\omega_j}$

11: **end for**

The dominant computational part is step 4 with a complexity of $\mathcal{O}(n^6)$.

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A JD-type method for two-real-param. EVP

Solve the NEP in two real parameters

$$\boxed{T(\omega, \tau)u := (i\omega M + A + e^{-i\omega\tau} B)u = 0}$$

by a straight forward adaption of (nonlinear) JD method:

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   known approximations of eigenvectors or random vector

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2. solve projected problem $V^H T(\omega,\tau)Vz = 0$
   use presented method for small problems

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3. compute approximate eigenvectors $u_i = Vz_i$ and residuals $r_i = T(\omega_i,\tau_i)u_i$
4. stop, if enough eigentriples have converged
   we use $\|r_i\|_2 \leq \mathrm{tol}$

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4. stop, if enough eigentriples have converged
5. compute a correction c of an approx. eigenvector $\hat{u}$ i.e., $\hat{u} + c$ is a better approximation
   that's the ineresting part

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6. expand $V \leftarrow \mathrm{orth}[V, c]$ and GOTO 2
   repeat as long as neccessary

Correction equation - medium sized problems

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Wanted: correction (δ, ϵ, c)

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- $$\begin{bmatrix} T(\hat{\omega}, \hat{\tau}) & T_\omega(\hat{\omega}, \hat{\tau})\hat{u} & T_\tau(\hat{\omega}, \hat{\tau})\hat{u} \\ \hat{u}^H & 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ \delta \\ \varepsilon \end{bmatrix} = \begin{bmatrix} -\hat{r} \\ 0 \end{bmatrix}.$$
  linear system, $n+1$ complex equations,
  n complex $+$ 2 real unknowns $\rightarrow$ no standard software available

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- $\begin{bmatrix} T(\hat\omega, \hat\tau) & T_\omega(\hat\omega, \hat\tau)\hat u & T_\tau(\hat\omega, \hat\tau)\hat u \\ \hat u^H & 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ \delta \\ \varepsilon \end{bmatrix} = \begin{bmatrix} -\hat r \\ 0 \end{bmatrix}.$

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Correction equation - medium sized problems

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Lemma: $d = \overline{c}, \delta, \epsilon \in \mathbb{R}$
What if n is large and a preconditioner is available for $\hat T$ only?

Correction equation - large size

With

$$\mathbb{T} = \begin{bmatrix} \hat{T} & 0 \\ 0 & \bar{\hat{T}} \end{bmatrix}, \quad K = \begin{bmatrix} \hat{T}_\omega \hat{u} & \hat{T}_\tau \hat{u} \\ \overline{\hat{T}_\omega \hat{u}} & \overline{\hat{T}_\tau \hat{u}} \end{bmatrix}, \quad U = \begin{bmatrix} \hat{u} & 0 \\ 0 & \bar{\hat{u}} \end{bmatrix}$$

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is complemented by projectors

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How does this help with the preconditioner?

Preconditioning the correction equation

Correction equation

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preconditioner:

$(I - K(U^HK)^{-1}U^H)\mathbb{P}(I - UU^H)$ with $\mathbb{P} = \begin{bmatrix} P & 0 \\ 0 & \bar{P} \end{bmatrix}$

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need to apply in a left preconditioned Krylov solver:

$$(I - UU^H)\mathbb{P}^{-1}(I - K(U^H\mathbb{P}^{-1}K)^{-1}U^H\mathbb{P}^{-1})(I - K(U^H K)^{-1}U^H)\mathbb{T}(I - UU^H)$$

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Efficient implementation needs one application of $P^{-1}\hat{T}$ per iteration and additionally 2 T products and 3 P solves as in other JD variants

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- if M, A, B are real, then eigentriples come in pairs (ω, τ, u), $(-\omega, \tau, \overline{u})$
- when an eigenvector u converges, add $\overline{u}$ to searchspace

Outline

Numerical example

Consider the parabolic problem

$$u_t - \nabla((1 + x^2 + y^2 + z^2)\nabla u) + [1, 0, -1]\nabla u + u - \alpha(1 + x^2)u(t - \tau) = 0 \quad (1)$$

with spatial variables x, y and z on $\Omega = (0, 1) \times (0, 1) \times (0, 1)$ with Dirichlet boundary condition $u = 0$ on $\partial\Omega$.

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A discretization with piecewise quadratic ansatz functions on a tetrahedral grid using COMSOL yielded an eigenvalue problem

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of dimension $n = 80623$.

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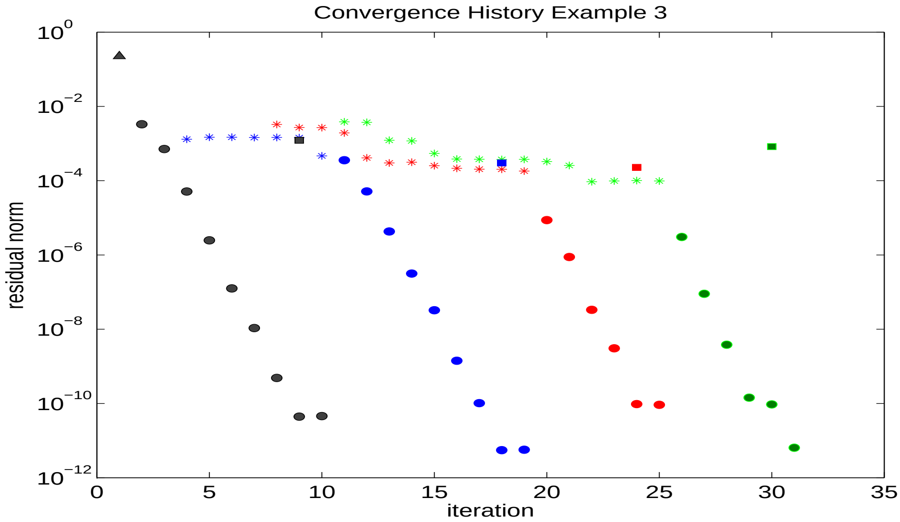
For $\alpha = 100$ the problem has four pairs of eigentriples.

Numerical Example



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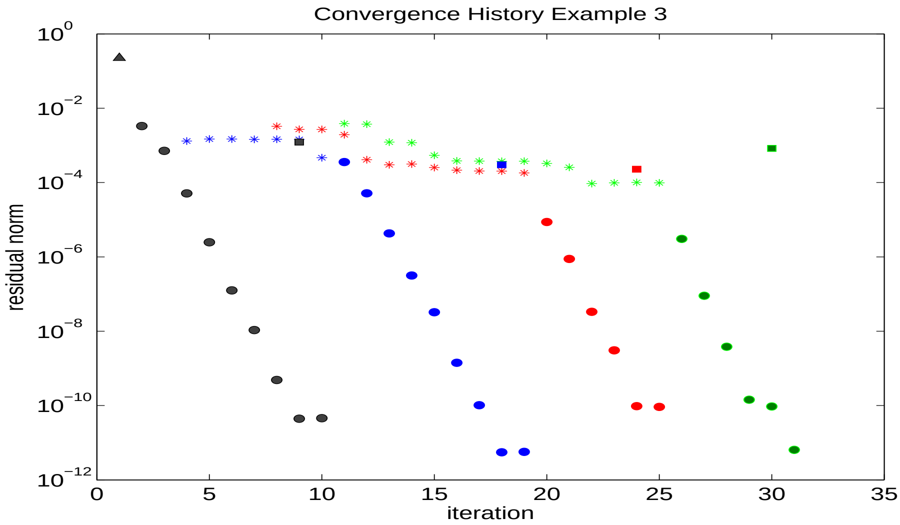
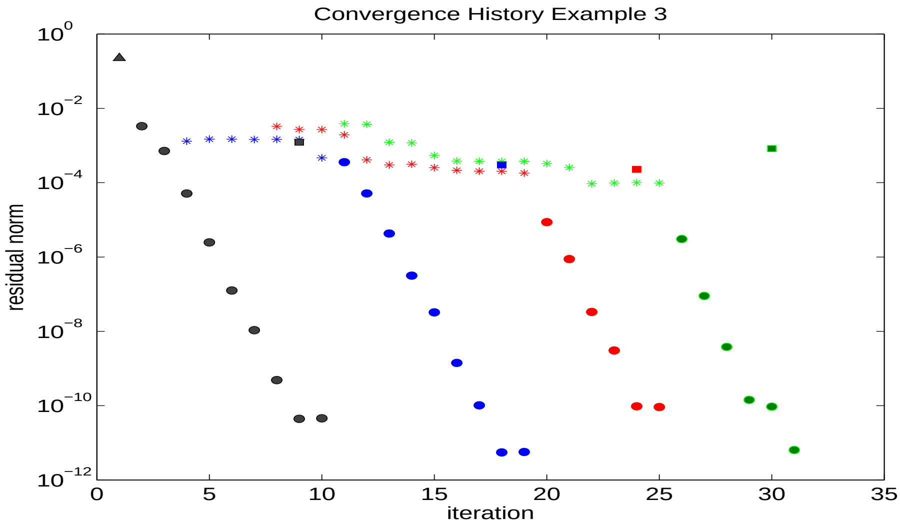


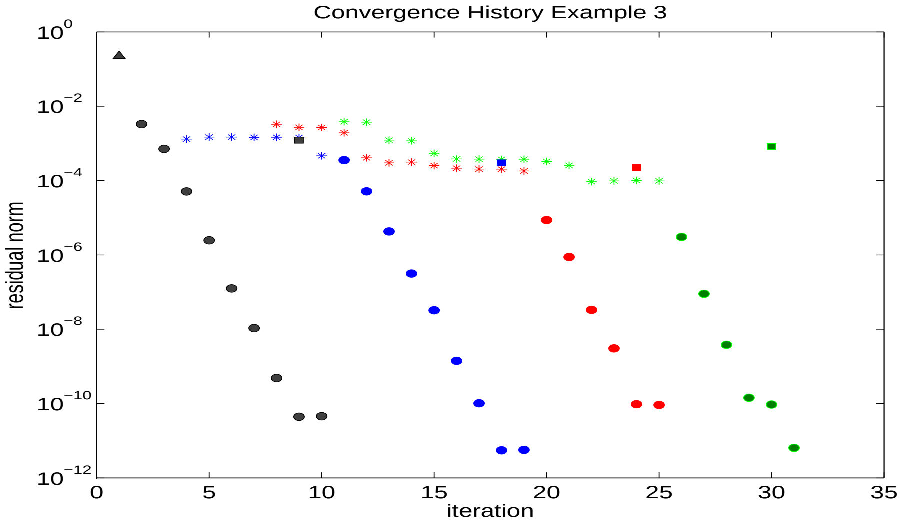
- x axis: number of iteration: tic=5; together 31

Numerical Example



- y axis: norm of residual: tic$=10^{-2}$; tolerance$=10^{-10}$

Numerical Example



- first iteration: alternative selection, then Ritz values

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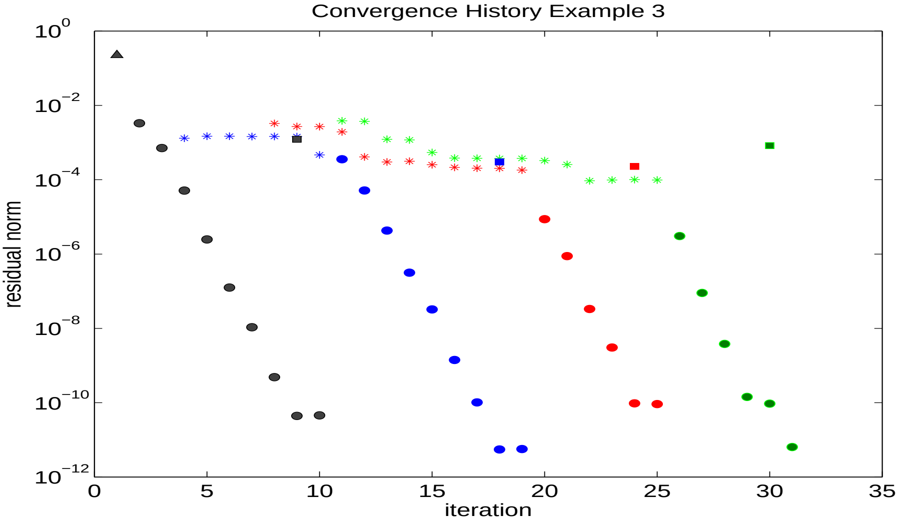
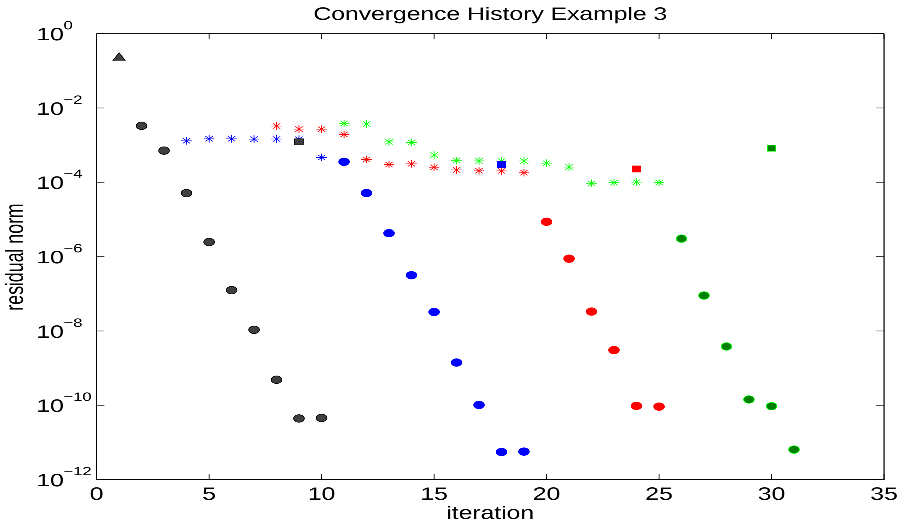


- linear system: `P=luinc(A,1e-3)`; GMRES terminated after at most 10 steps or if residual was reduced by 10^{-1}

Numerical Example



- first converged triple: 9 iterations, then expand by $\bar{u}$

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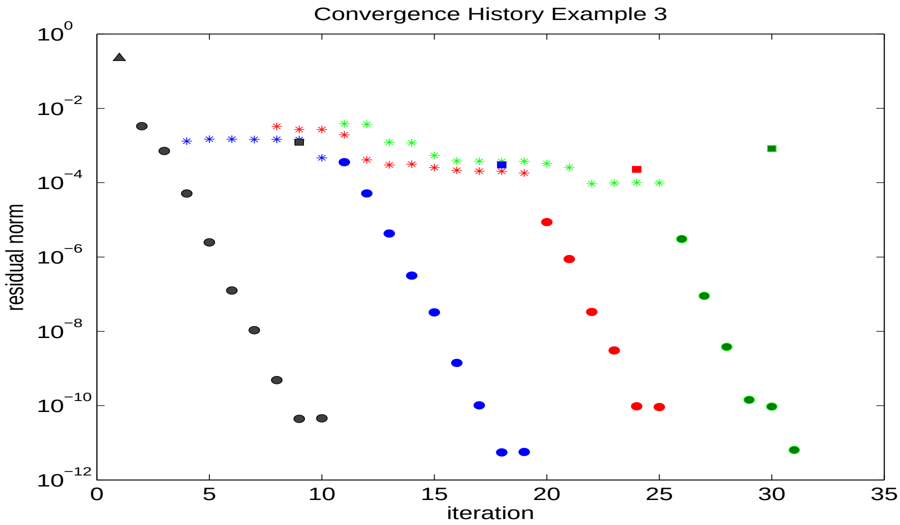
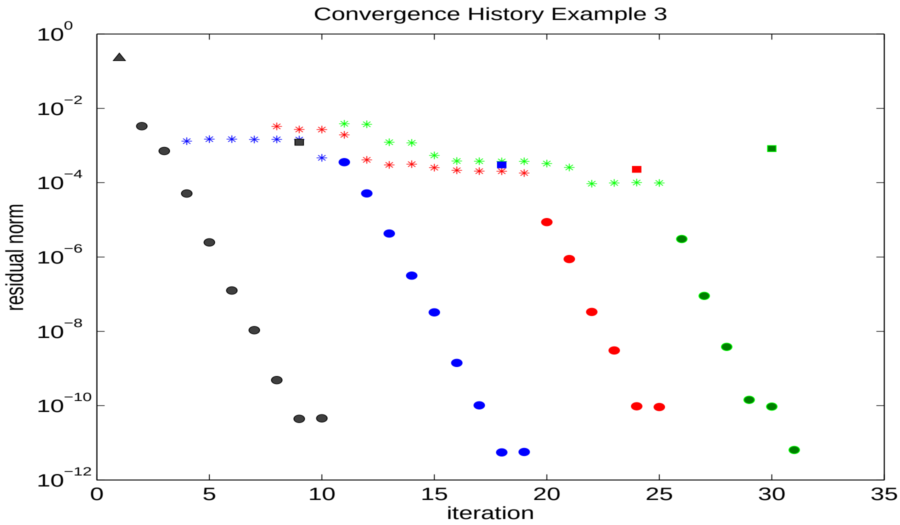


- then $8+1$, $5+1$, $5+1$ iterations for further triple pairs

Numerical Example



- robust (lots of runs with random start vector)

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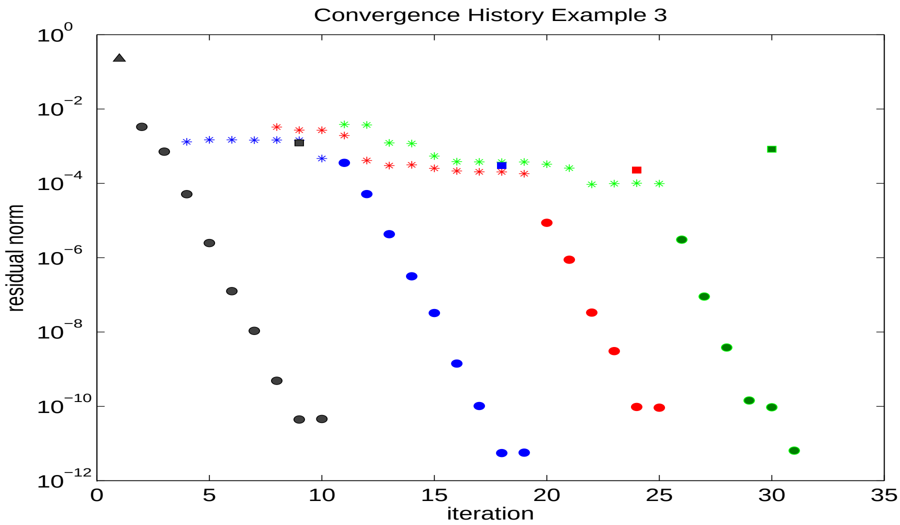

- restarts after 12, 16, 21, 25, 28 iterations, no slowdown

Conclusion

$$\boxed{(i\omega M + A + e^{i\omega\tau} B)u = T(\omega, \tau)u = 0}$$

- two real instead of one complex parameter
- "direct" method for small critical delay problems
- robust JD variant for larger problems

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References

- K. Meerbergen, C. Schröder, and H. Voss
A Jacobi-Davidson method for two real parameter nonlinear eigenvalue problems arising from delay differential equations,
Report 148, Institut f. Numerische Simulation, TU Hamburg-Harburg, October 2010, submitted.

Thanks for your attention!