

On Sylvester's Law of Inertia for Nonlinear Eigenvalue Problems

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Sylvester's law of inertia

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This leads to

- Localization of eigenvalues
- Bisection method and secant method for eigenvalues
- Determine initial approximation for faster eigensolvers
- Test whether eigenvalues in a given interval are missing

- 1 Minmax Characterization
- 2 Sylvester's law for nonlinear eigenproblems
- 3 Hyperbolic eigenvalue problems
- 4 Nonoverdamped quadratic eigenproblems
- 5 Gyroscopically stabilized systems

Outline

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Nonlinear Eigenvalue Problem

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Problems of this type arise in damped vibrations of structures, conservative gyroscopic systems, lateral buckling problems, problems with retarded arguments, fluid-solid vibrations, and quantum dot heterostructures, e.g.

Nonlinear minmax theory

Assume that for fixed $x \in \mathbb{C}^n$, $x \neq 0$ the real equation

$$f(\lambda, x) := x^H T(\lambda)x = 0$$

has at most one solution $\lambda =: \rho(x)$ in J .

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Then equation $f(\lambda, x) = 0$ implicitly defines a functional p on some subset D of \mathbb{C}^n which we call the **Rayleigh functional**.

Let

$$(\lambda - p(x))f(\lambda, x) > 0 \quad \text{for every } \lambda \neq p(x) \text{ and every } x \in D.$$

Overdamped problems

If p is defined on $D = \mathbb{C}^n \setminus \{0\}$ then the problem $T(\lambda)x = 0$ is called **overdamped**.

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Theorem (Duffin 1955, Rogers 1964)

Under the conditions above an overdamped problem has exactly n eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ which can be characterized by

$$\lambda_j = \min_{\dim V=j} \max_{x \in V \setminus \{0\}} p(x).$$

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Then all conditions are satisfied, ρ is the restriction of the Rayleigh quotient R_A to

$$D := \{x \neq 0 : R_A(x) \in J\},$$

and $\inf_{x \in D} \rho(x)$ will in general not be an eigenvalue.

Enumeration of eigenvalues

If $\lambda \in J$ is an eigenvalue of $T(\cdot)$ then $\mu = 0$ is an eigenvalue of the linear problem $T(\lambda)y = \mu y$, and therefore there exists $\ell \in \mathbb{N}$ such that

$$0 = \max_{V \in H_\ell} \min_{v \in V \setminus \{0\}} \frac{v^H T(\lambda) v}{\|v\|^2}$$

where H_ℓ denotes the set of all ℓ -dimensional subspaces of \mathbb{C}^n .

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In this case λ is called an **ℓ -th eigenvalue of $T(\cdot)$** .

Minmax characterization (V., Werner 1982, V. 2009)

Under the conditions given above it holds:

- (i) For every $\ell \in \mathbb{N}$ there is at most one ℓ -th eigenvalue of $T(\cdot)$ which can be characterized by

$$\lambda_\ell = \min_{V \in H_\ell, V \cap D \neq \emptyset} \sup_{v \in V \cap D} p(v). \quad (*)$$

The set of eigenvalues of $T(\cdot)$ in J is at most countable.

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- (ii) If

$$\lambda_\ell := \inf_{V \in H_\ell, V \cap D \neq \emptyset} \sup_{v \in V \cap D} \rho(v) \in J$$

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- (iii) $\tilde{\lambda}$ is an ℓ -th eigenvalue if and only if $\mu = 0$ is the ℓ largest eigenvalue of the linear eigenproblem $T(\tilde{\lambda})x = \mu x$.

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- (iii) $\tilde{\lambda}$ is an ℓ -th eigenvalue if and only if $\mu = 0$ is the ℓ largest eigenvalue of the linear eigenproblem $T(\tilde{\lambda})x = \mu x$.
- (iv) The minimum in (*) is attained for the invariant subspace of $T(\lambda_\ell)$ corresponding to its ℓ largest eigenvalues.

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For $\sigma \in J$ let (π, ν, ζ) be the inertia of $T(\sigma)$. Then the nonlinear eigenproblem $T(\lambda)x = 0$ has π eigenvalues that are smaller than σ , ν eigenvalues that exceed σ , and σ is an eigenvalue of multiplicity ζ .

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Sketch of proof: Show that

- (i) If W denotes the invariant subspace of $T(\sigma)$ corresponding to its positive eigenvalues, then $f(\sigma, x) = x^H T(\sigma)x > 0$ for $x \in W$, $x \neq 0$. Hence, $\rho(x) < \sigma$, and therefore

$$\lambda_\pi = \min_{\dim V=\pi} \max_{x \in V} \rho(x) \leq \max_{x \in W} \rho(x) < \sigma.$$

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- (ii) For every subspace V with $\dim V = \pi + \zeta + 1$ there exists $x \in V$ with $\rho(x) > \sigma$, and thus

$$\lambda_{\pi+\zeta+1} = \min_{\dim V = \pi + \zeta + 1} \max_{x \in V} \rho(x) > \sigma.$$

Sylvester's law for non-overdamped problems

Extreme eigenvalues:

Assume that $T : J \rightarrow \mathbb{C}^{n \times n}$ satisfies the conditions of the minmax characterization, and assume that there exists $\mu \in J$ such that $T(\mu)$ is negative definite.

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For $\sigma \in J$, $\sigma > \mu$ let (π, ν, ζ) be the inertia of $T(\sigma)$. Then the nonlinear eigenproblem $T(\lambda)x = 0$ has exactly π eigenvalues in J that are smaller than σ .

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General case:

Assume that for every ρ dimensional subspace V with $V \cap D \neq \emptyset$ there exists $x \in V \cap D$ with $\rho(x) > \mu$, $\mu \in J$.

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Hyperbolic quadratic pencils

A quadratic matrix polynomial

$$Q(\lambda) := \lambda^2 A + \lambda B + C, \quad A = A^H > 0, \quad B = B^H, \quad C = C^H$$

is **hyperbolic** if for every $x \in \mathbb{C}^n$, $x \neq 0$ the quadratic polynomial

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$$p_{\pm}(x) = -\frac{x^H B x}{2x^H A x} \pm \sqrt{\left(\frac{x^H B x}{2x^H A x}\right)^2 - \frac{x^H C x}{x^H A x}}.$$

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The ranges $J_{\pm} := p_{\pm}(\mathbb{C}^n \setminus \{0\})$ are disjoint real intervals with $\max J_- < \min J_+$

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(Q, J_+) and $(-Q, J_-)$ satisfy the conditions of the variational characterization of eigenvalues, i.e. there exist $2n$ eigenvalues (cf. Duffin 1955)

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$$\lambda_j = \min_{\dim V=j} \max_{x \in V, x \neq 0} p_-(x) \quad \text{and} \quad \lambda_{n+j} = \min_{\dim V=j} \max_{x \in V, x \neq 0} p_+(x), \quad j = 1, \dots, n.$$

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For $\sigma \in \mathbb{R}$ and $x \neq 0$ with $f(\sigma, x) := x^H Q(\sigma)x > 0$ and $2\sigma x^H Ax + x^H Bx < 0$ it follows that $p_-(x) > \sigma$, and therefore $\sigma \leq \lambda_n$. Similarly, $f(\sigma, x) := x^H Q(\sigma)x > 0$ and $2\sigma x^H Ax + x^H Bx > 0$ implies $\sigma > \lambda_{n+1}$.

Sylvester's law for hyperbolic quadratic pencils

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- (2) For $\pi = 0$ and $\zeta > 0$ let $x \neq 0$ be an element of the null space of $Q(\sigma)$. If $2\sigma x^H M x + x^H C x < 0$ then $Q(\lambda)x = 0$ has $n - \zeta$ eigenvalues in $(-\infty, \sigma)$, n eigenvalues in (σ, ∞) and $\sigma = \lambda_n$ with multiplicity ζ , and if $2\sigma x^H M x + x^H C x > 0$ then $Q(\lambda)x = 0$ has n eigenvalues in $(-\infty, \sigma)$, $n - \zeta$ eigenvalues in (σ, ∞) and $\sigma = \lambda_{n+1}$ with multiplicity ζ .

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- (3) For $\pi > 0$ and $\zeta = 0$ let $x \neq 0$ such that $f(\sigma; x) > 0$. If $2\sigma x^H Mx + x^H Cx < 0$ then $Q(\lambda)x = 0$ has $n - \pi$ eigenvalues in $(-\infty, \sigma)$ and $n + \pi$ eigenvalues in (σ, ∞) , and if $2\sigma x^H Mx + x^H Cx > 0$ then $Q(\lambda)x = 0$ has $n + \pi$ eigenvalues in $(-\infty, \sigma)$ and $n - \pi$ eigenvalues in (σ, ∞) .

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- (4) For $\pi > 0$ and $\zeta > 0$ let $x \neq 0$ such that $f(\sigma; x) > 0$. If $2\sigma x^H Mx + x^H Cx < 0$ then $Q(\lambda)x = 0$ has $n - \pi - \zeta$ eigenvalues in $(-\infty, \sigma)$ and $n + \pi$ eigenvalues in (σ, ∞) , and if $2\sigma x^H Mx + x^H Cx > 0$ then $Q(\lambda)x = 0$ has $n + \pi$ eigenvalues in $(-\infty, \sigma)$ and $n - \pi - \zeta$ eigenvalues in (σ, ∞) . In either case σ is an eigenvalue with multiplicity ζ .

hyperbolic pencils

Let

$$P(\lambda) = \sum_{j=0}^k \lambda^j A_j, \quad A_j = A_j^H, \quad j = 0, \dots, k, \quad A_k > 0$$

be hyperbolic, i.e. $f(\lambda; x) := x^H P(\lambda) x = 0$ has exactly k real roots for $x \neq 0$

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Then there exist k disjoint open intervals $\Delta_j \subset \mathbb{R}, j = 1, \dots, k$ such that $P(\lambda)x = 0$ has exactly n eigenvalues in each Δ_j which allow for a minmax characterization. To fix the numeration let $\sup \Delta_j < \inf \Delta_{j-1}$.

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For $\sigma \in \mathbb{R}$ let (π, ν, ζ) be the inertia of $T(\sigma)$, and let $x \in \mathbb{C}^n$ such that $x^H T(\sigma)x > 0$.

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For $\sigma \in \mathbb{R}$ let (π, ν, ζ) be the inertia of $T(\sigma)$, and let $x \in \mathbb{C}^n$ such that $x^H T(\sigma)x > 0$.

If $f(\cdot; x)$ has exactly j roots which exceed σ then it holds that

$$\sigma \in \Delta_{j+1} \quad \text{or} \quad \sigma \in [\sup \Delta_{j+1}, \inf \Delta_j] \quad \text{or} \quad \sigma \in \Delta_j.$$

hyperbolic pencils

Let

$$P(\lambda) = \sum_{j=0}^k \lambda^j A_j, \quad A_j = A_j^H, \quad j = 0, \dots, k, \quad A_k > 0$$

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The inertia and the derivative $\frac{\partial}{\partial \lambda} f(\sigma; x)$ yield the localization of σ within the spectrum of $T(\lambda)x = 0$.

Outline

- 1 Minmax Characterization
- 2 Sylvester's law for nonlinear eigenproblems
- 3 Hyperbolic eigenvalue problems
- 4 Nonoverdamped quadratic eigenproblems
- 5 Gyroscopically stabilized systems

Nonoverdamped quadratic eigenproblems

$$Q(\lambda) = \lambda^2 A + \lambda B + C, \quad A > 0, B > 0, C > 0$$

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Let

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and $J_- := (-\infty, \delta_+)$, $J_+ = (\delta_-, \infty)$.

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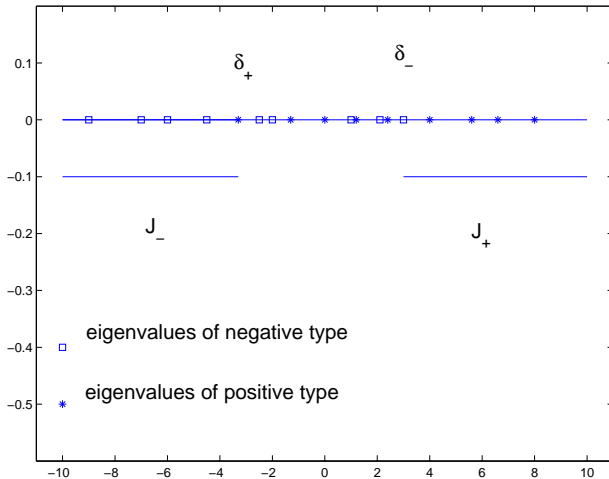
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Hence, for $\sigma < \delta_+$ with $\text{in}(T(\sigma)) = (\pi, \nu, \zeta)$ there are ν eigenvalues of $T(\lambda)x = 0$ in $(-\infty, \delta_+)$, and for $\sigma \in (0, \delta_-)$ with $\text{in}(T(\sigma)) = (\pi, \nu, \zeta)$ there are ν eigenvalues of $T(\lambda)x = 0$ in $(\delta_-, 0)$

Nonoverdamped quadratic eigenproblems



Nonoverdamped quadratic eigenproblems

Theorem

Let $A, B, C > 0$. Then it holds that

$$\tilde{\delta}_+ := -\sqrt{\max_{x \neq 0} \frac{x^H C x}{x^H A x}} \leq \delta_+ = \inf\{p_+(x) : p_+(x) \in \mathbb{R}\}$$

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Nonoverdamped quadratic eigenproblems

Proof

$$f(p_+(x); x) = x^H Q(p_+(x)) x = 0 \iff x^H B x = -p_+(x) x^H A x - \frac{1}{p_+(x)} x^H C x.$$

Nonoverdamped quadratic eigenproblems

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$$f(p_+(x); x) = x^H Q(p_+(x)) x = 0 \quad \Longleftrightarrow \quad x^H B x = -p_+(x) x^H A x - \frac{1}{p_+(x)} x^H C x.$$

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$$\frac{\partial}{\partial \lambda} f(p_+(x); x) = 2p_+(x) x^H A x + x^H B x = p_+(x) x^H A x - \frac{1}{p_+(x)} x^H C x \geq 0$$

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Eliminating $x^H A x$ one gets $-2\lambda_{\max}(C, B) \leq \delta_+ \leq \delta_- \leq -2\lambda_{\min}(C, B).$

Example

$$A = \text{eye}(20); \quad B = \text{randn}(20); \quad B = B' * B; \quad C = \text{randn}(20); \quad C = C' * C;$$

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12 eigenvalues of negative type are less than $-\sqrt{\max(\lambda(C, A))}$ and 6 eigenvalues of positive type exceed $-\sqrt{\min(\lambda(C, A))}$.

Outline

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Gyroscopically stabilized pencils

A quadratic matrix polynomial

$$Q(\lambda) := \lambda^2 I + \lambda B + C, \quad B = B^H, \det B \neq 0, C = C^H > 0$$

is **gyroscopically stabilized** if for some $k > 0$ it holds that

$$|B| > kI + k^{-1}C$$

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Motivation: For $G^H = -G$ the free motions of a conservative, time-invariant linear system oscillating about an unstable equilibrium under action of a gyroscopic force are governed by

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Making the substitution $u(t) = x \exp(\mu t)$ with x independent of t , and then the rotation of the parameter $\lambda = -i\mu$ leads to the eigenvalue problem $Q(\lambda)x = 0$ where $B = iG$ is clearly indefinite.

Gyroscopically stabilized pencils ct.

Theorem (Barkwell, Lancaster, Markus 1992)

- The spectrum of a gyroscopic stabilized pencil is real, i.e. Q is quasihyperbolic.

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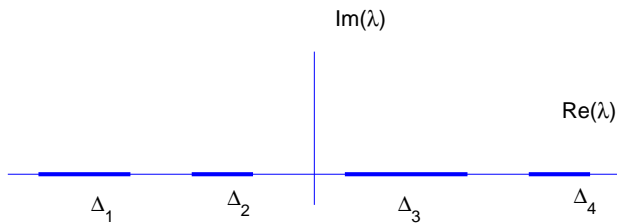
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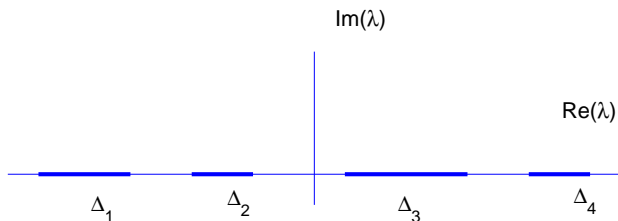
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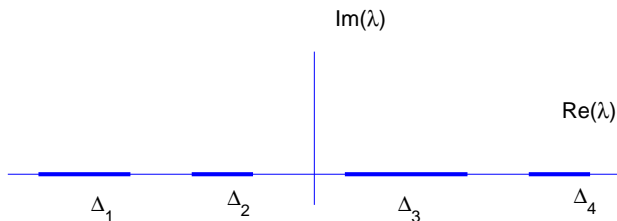
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- The 2ν positive eigenvalues lie in two disjoint intervals, ν eigenvalues in each; the ones in the left interval are of negative type, the ones in the right interval are of positive type.





- in Δ_1 there are π eigenvalues of negative type
- in Δ_2 there are π eigenvalues of positive type
- in Δ_3 there are $n - \pi$ eigenvalues of negative type
- in Δ_4 there are $n - \pi$ eigenvalues of positive type



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There is a basis of eigenvectors corresponding to eigenvalues in $\Delta_1 \cup \Delta_3$, and there is basis of eigenvectors corresponding to eigenvalues in $\Delta_2 \cup \Delta_4$.

Gyroscopically stabilized pencils ct.

Theorem (Lancaster, Markus, Zhou 2003)

- There exists a π dimensional subspace V of $\mathbb{C}^{n \times n}$ such that $x^H Q(-k)x < 0$ for every $x \in V \setminus \{0\}$.
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Since $\lambda \mapsto x^H Q(\lambda)x$ is a parabola and $x^H Q(0)x > 0$ it follows at once $V \cap W = \{0\}$, i.e.

$$\dim(V + W) = n$$

Minmax characterization

Let

$$p_{-}^{+}(x) := \begin{cases} p_{-}(x) & \text{if } p_{-}(x) > 0 \\ \infty & \text{else} \end{cases}$$

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If

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From these variational characterizations of eigenvalues one obtains a Sylvester Theorem for gyroscopically stabilized quadratic eigenvalue problems in an obvious way.

Minmax for Gyroscopically Stabilized Pencils

Lemma

For $x \neq 0$ let $f(\lambda; x) := x^H Q(\lambda)x = 0$ and $\lambda \geq \sqrt{\lambda_{\max}(C)}$. Then it holds that $x^H Q'(\lambda)x \geq 0$.

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Proof Similar to the nonoverdamped case.

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In $(\sqrt{\lambda_{\max}(\mathbf{C})}, \infty)$ the conditions of the minmax characterization with Rayleigh functional p_+^+ are satisfied. Hence there are at most $n - \pi$ eigenvalues which are maxmin values of p_+^+ . Hence, if $\sigma \geq \sqrt{\lambda_{\max}(\mathbf{C})}$ and $\text{in}Q(\sigma) = (\pi, \nu, 0)$, then there are exactly ν eigenvalues of $Q(\lambda)x = 0$ in (σ, ∞) .

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Similar results hold for the intervals $(-\infty, -\sqrt{\lambda_{\max}(C)})$, $(-\sqrt{\lambda_{\min}(C)}, 0)$, and $(0, \sqrt{\lambda_{\min}(C)})$, and localization of eigenvalues via a Sylvester law is possible in these intervals as well.