Verified Linear Programming – a Comparison

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Why linear programming?

- most basic form of optimization
- wide range of applications:
  oil refinery problems, flap settings on aircraft, industrial production and allocation, image restoration, linearization, linear relaxations in global optimization, ...

Lovasz, 1980

If one would take statistics about which mathematical problem is using up most of the computer time in the world, then (not including database handling problems like sorting and searching) the answer would probably be linear programming.

- Ordóñez and Freund, 2003:
  71% of Netlib lp problems are ill-posed

Ben-Tal and Nemirovski, 2000

In real-world applications of Linear Programming one cannot ignore the possibility that a small uncertainty in the data (intrinsic for most real-world LP programs) can make the usual optimal solution of the problem completely meaningless from a practical viewpoint.

Outline

- iCOs, RealPaver
  - constraint satisfaction codes
  - no optimization $\Rightarrow$ lower bound from non-satisfiable problems
  - iCOs: constraint programming, interval analysis
  - RealPaver: branch-and-prune, interval Newton method

Software to solve LP rigorously

- iCOs, RealPaver
  - constraint satisfaction codes
  - no optimization $\Rightarrow$ lower bound from non-satisfiable problems
  - iCOs: constraint programming, interval analysis
  - RealPaver: branch-and-prune, interval Newton method
- perPlex
  - rational arithmetic $\Rightarrow$ requires rational solution
  - cannot handle interval data
  - proof of concept $\Rightarrow$ just verifies, doesn't compute solution

More software to solve LP rigorously

- GlobSol, Numerica, COSY
  - global optimization
  - GlobSol: interval branch-and-bound, automatic differentiation, constraint propagation, interval Newton, ...
  - Numerica: interval methods (Hansen–Sengupta), constraint satisfaction
  - COSY: branch-and-bound, Taylor model arithmetic

More software to solve LP rigorously

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  - global optimization
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  - Numerica: interval methods (Hansen–Sengupta), constraint satisfaction
- Lurupa
  - verified linear programming
  - algorithms developed by Jansson at IRC
  - iteratively enclose near optimal feasible points

A proper test set

- 99 random problems after Rosen and Suzuki
  - choose solution $\Rightarrow$ generate problem from KKT
  - non–degenerate
  - exactly known, integral solution
  - 3 instances per dimension set (variables, inequalities, equations)
- 19 real-world problems from Netlib and Meszaros's collection
  - less than 50 variables
  - large finite search region
- 1 hour timeout

Results for random problems

5 variables (total 36)

	solved	t_{max}
iCOs	36	0.091s
RealPaver	34	1378.100s
perPlex	36	0.028s
GlobSol	36	548.582s
Lurupa	36	$<$ 0.010s

Numerica solved 5 inequalitites and 5 variables in 326.7s

Results for random problems

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10 variables (total 36)

	solved	t_{max}
iCOs	36	0.254s
RealPaver	0	
perPlex	36	0.036s
GlobSol	3	666.963s
Lurupa	36	$<$ 0.010s

Results for larger random problems

25 variables (total 18)

	solved	t_{max}
iCOs	5	0.273s
perPlex	18	0.077s
Lurupa	18	$<$ 0.010s

iCOs

- solved problems with 15 or 25 equations
- ran out of memory for the remaining problems

Results for larger random problems

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iCOs

- solved problems with 15 or 25 equations
- ran out of memory for the remaining problems

500 variables (total 9)

	solved	t_{max}
perPlex	0	
Lurupa	9	29.2s

Results for real-world problems

feasible problems (total 15)

	solved	t_{max}
iCOs	7	0.401s
RealPaver	12	207.500s
perPlex	15	0.021s
GlobSol	6	4.564s
Lurupa	15	$<$ 0.010s

iCOs wronlgy claims 3 problems to be infeasible

Results for real-world problems

feasible problems (total 15)

	solved	t_{max}
iCOs	7	0.401s
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iCOs wronlgy claims 3 problems to be infeasible

infeasible problems (total 4)

	terminated	t_{max}
iCOs	4	1.190s
RealPaver	4	< 0.010s
perPlex	0	
GlobSol	0	
Lurupa	4	0.040s

Lurupa returns some wide bounds ($[0, \infty]$, $[-\infty, \infty]$)

1. Introduction
2. Numerical Experience
   - Solvers
   - Test set
   - Results
3. Summary

Summary

- General optimization software can verify solutions up to $\sim 10 - 20$ variables
- Larger dimensions require to exploit special structure
- Lurupa implements appropriate algorithms and solves medium-scale problems in reasonable time

Dimensions for real-world problems

Meszaros's misc problems

	m	p	n
kleemin3	3	0	3
kleemin4	4	0	4
kleemin5	5	0	5
kleemin6	6	0	6
kleemin7	7	0	7
kleemin8	8	0	8
farm	5	2	12
p0033	15	0	33
refine	29	0	33
p0040	23	0	40
problem	0	12	46

Meszaros's infeas problems

	m	p	n
itest2	9	0	4
galenet	6	2	8
itest6	9	2	8
bgprtr	6	14	35

Netlib problems

	m	p	n
afiro	19	8	32
kb2	27	16	41
sc50a	30	20	48
sc50b	30	20	48

Meszaros's misc problems

	iCOs		RealPaver		perPlex	GlobSol	Lurupa
	t	Δ	t	Δ			
kleemin3	0.004	1e-12	0	1e-12	0.011	0.032	< 0.010
kleemin4	0.007	1e-12	0.030	1e-12	0.012	0.089	< 0.010
kleemin5	0.009	1e-12	0.210	1e-12	0.014	0.240	< 0.010
kleemin6	?		3.030	1e-12	0.012	0.639	< 0.010
kleemin7	?		207.500	1e-12	0.011	1.675	< 0.010
kleemin8	?		10.380	1e-4	0.012	4.564	< 0.010
farm	0.009	1e-12	1.480	1e-8	0.012	t	< 0.010
p0033	oom		t	1e-1	0.017	t	< 0.010
refine	oom		0	1e-1	0.021	e/t	< 0.010
p0040	oom		t	1e-1	0.014	t	< 0.010
problem	0.019	1e-12	0	1e-12	0.015	t	< 0.010

More detailed results for real-world problems

Meszaros's infeas problems

	iCOs	RealPaver	perPlex	GlobSol	Lurupa
itest2	0.947	0	lpinf	e/t	< 0.010s $[0, \infty]$
galenet	0	0	lpinf	e	< 0.010s $[0, \infty]$
itest6	1.190	0	lpinf	e/t	< 0.010s $[1.9e6, \infty]$
bgprtr	0.025	0	lpinf	t	0.040s $[-\infty, +\infty]$

Netlib problems

	iCOs		RealPaver		perPlex	GlobSol	Lurupa
	t	Δ	t	Δ			
afiro	oom		55.680	1e-12	0.012	t	< 0.01
kb2	oom		t	1e-1	0.016	t	< 0.01
sc50a	0.401	1e-12	157.700	1e-2	0.016	t	0.01
sc50b	0.342	1e-12	0.740	1e-1	0.016	t	< 0.01