When does $\|f(A)\| = f(\|A\|)$ hold true?

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ABSTRACT
Let a strongly stable norm $\|\cdot\|$ on the set M_n of complex n-by-n matrices be given, which means that $\|A^k\| \le \|A\|^k$ for all $A \in M_n$ and all $k = 1, 2, \dots$. Furthermore, let $f(x) = \sum_{k=0}^{\infty} c_k x^k$ be a power series with nonnegative coefficients $c_k \ge 0$ and radius of convergence $R > 0$. If $\|I\| > 1$, we additionally suppose that $c_0 = f(0) = 0$.

We aim to characterize those A with $\|A\| < R$ which fulfill $\|f(A)\| = f(\|A\|)$. We first show how to reduce the discussion of f to Neumann series. For matrix norms induced by uniformly convex vector norms, like the ℓ^p-norms, $p \in (1, \infty)$, it follows from known results on the Daugavet equation $\|I + A\| = 1 + \|A\|$ that $\|f(A)\| = f(\|A\|)$ holds true if, and only if, $\|A\|$ is an eigenvalue of A, provided that $c_k c_{k+1} \ne 0$ for some $k \ge 0$. Under adapted assumptions on the c_k we prove that this equivalence remains true for the ℓ^1- and the ℓ^∞-norm, for unitarily invariant matrix norms, and for the numerical radius. We conjecture this equivalence to be valid for all strongly stable norms if $c_k > 0$ for all $k \ge 1$.

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1. Introduction

For $n \in \mathbb{N} = \{1, 2, \dots\}$ let M_n denote the set of complex n-by-n matrices and $I = I_n \in M_n$ the n-by-n identity matrix. Let a *strongly stable* norm $\|\cdot\|$ on M_n be given (see [5]), which means that

$$\|A^k\| \le \|A\|^k \quad \text{for all } A \in M_n \text{ and all } k \in \mathbb{N}. \tag{1}$$

If, in addition, $\|\cdot\|$ is submultiplicative, i.e., if $\|XY\| \le \|X\|\|Y\|$ for all $X, Y \in M_n$, then $\|\cdot\|$ is called a *matrix norm*. One of the most prominent examples of a strongly

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stable norm which is not a matrix norm is the numerical radius

$$r(A) := \max\{|u^*Au| \mid u \in \mathbb{C}^n, u^*u = 1\},$$

where u^* denotes the conjugate transpose of $u \in \mathbb{C}^n$. If

$$f(x) = \sum_{k=0}^{\infty} c_k x^k$$

is a power series with all $c_k \geq 0$ and radius of convergence $R > 0$, then the triangle inequality and (1) readily imply

$$\|f(A)\| = \|\sum_{k=0}^{\infty} c_k A^k\| \leq c_0\|I\| + \sum_{k=1}^{\infty} c_k\|A\|^k = f(0)(\|I\| - 1) + f(\|A\|) \tag{2}$$

for all $A \in M_n$ with $\|A\| < R$. Note that (1) applied to $A := I$ and $k := 2$ yields $\|I\| = \|I^2\| \leq \|I\|^2$ so that $\|I\| \geq 1$. However, $\|I\| > 1$ may hold true as, for example, for the Frobenius norm in dimension $n \geq 2$. Thus, the unpleasant addend $f(0)(\|I\|-1)$ on the right-hand side in (2) disappears if, and only if,

$$f(0) = 0 \quad \text{or} \quad \|I\| = 1, \tag{3}$$

in which case we have

$$\|f(A)\| \leq f(\|A\|). \tag{4}$$

Under the assumption (3), which shall hold true in what follows, we are interested in determining those matrices A (with $\|A\| < R$) for which equality holds true in (4). To do so, first note that [6, Theorem 5.7.10] and (1) imply

$$\rho(X) = \lim_{k\to\infty} \|X^k\|^{1/k} \leq \|X\| \qquad \text{for all } X \in M_n \tag{5}$$

where $\rho(X)$ is the spectral radius of X. For $\operatorname{spec}(X)$ denoting the spectrum of $X \in M_n$, we have $\operatorname{spec}(f(A)) = f(\operatorname{spec}(A))$. Now, if $\|A\| \in \operatorname{spec}(A)$, then (4), (5), and $c_k \geq 0$ for all k yield

$$f(\|A\|) \geq \|f(A)\| \geq \rho(f(A)) = \max\{|f(\lambda)| \mid \lambda \in \operatorname{spec}(A)\} \geq |f(\|A\|)| = f(\|A\|),$$

so that $\|f(A)\| = f(\|A\|)$. This simple observation is recorded as

Remark 1. If $\|\cdot\|$ is strongly stable and $\|A\|$ is an eigenvalue of A, then equality holds true in (4) provided that (3) is satisfied.

In general, the converse is not true even for $n = 1$, where, for example, $f(x) := x^2$ and $A := -1$ fulfill $|f(A)| = 1 = f(|A|)$, but $\rho(A) = 1$ is not an eigenvalue of A. Thus, in order to turn Remark 1 into an equivalence statement, which is the task of this paper, further conditions must be imposed on $f(x)$ or $\|\cdot\|$. We conjecture:

Conjecture 1.1. If $f^{(k)}(0) > 0$ for all $k \in \mathbb{N}$ and $f(0)(\|I\| - 1) = 0$, then $\|f(A)\| = f(\|A\|)$ holds true if, and only if, $\|A\|$ is an eigenvalue of A.

We could neither prove this conjecture for general strongly stable norms, nor could we find a counterexample. In this note we prove the following:

Theorem 1.2. *Let $\|\cdot\|$ be a norm on the set M_n of complex n-by-n matrices. Furthermore, let $f(x) := \sum_{k=0}^{\infty} c_k x^k$ with all $c_k \geq 0$, radius of convergence $R \in (0, \infty]$, and $f(0)(\|I\| - 1) = 0$. Suppose that one of the following cases holds true: the norm is*

1) *induced by a uniformly convex vector norm*[1] *and $c_k c_{k+1} \neq 0$ for some $k \geq 0$,*
2) *unitarily invariant*[2] *and $c_k c_{k+1} \neq 0$ for some $k \geq 0$,*
3) *the numerical radius and $c_k c_{k+1} \neq 0$ for some $k \geq 1$,*
4) *the ℓ^1- or the ℓ^∞-norm and $c_k \neq 0$ for all $k \geq k_0$ and some $k_0 \geq 0$.*

Then, for $A \in M_n$ with $\|A\| < R$, we have $\|f(A)\| = f(\|A\|)$ if, and only if, $\|A\|$ is an eigenvalue of A.

Before the proof we give some general remarks. We first show how to reduce the discussion of power series to Neumann series or even to $f(x) = x^j(1 + x)$ for some $j \geq 0$. Then, part 1) for uniformly convex vector norms is a consequence of a result by Abramovich et al. [1] on the so-called Daugavet equation $\|I + A\| = 1 + \|A\|$.

Theorem 1.3 (Abramovich et al.)**.** *If $\|\cdot\|$ is induced by a uniformly convex vector norm, then the Daugavet equation $\|I + A\| = 1 + \|A\|$ holds true for an $A \in M_n$ if, and only if, $\|A\|$ is an eigenvalue of A.*

Recall that for $1 < p < \infty$ the ℓ^p-norms $\|x\|_p := (\sum_{k=1}^{n} |x_k|^p)^{1/p}$, $x \in \mathbb{C}^n$, are uniformly convex so that the induced matrix norms

$$\|A\|_p := \max\{\|Ax\|_p \mid \|x\|_p = 1,\ x \in \mathbb{C}^n\}, \quad A \in M_n$$

fulfill the assumptions of Theorem 1.3. Contrary, the ℓ^1-norm $\|x\|_1 := \sum_{k=1}^{n} |x_k|$ and the ℓ^∞-norm $\|x\|_\infty = \max_{1 \leq k \leq n} |x_k|$ are not uniformly convex, so that their induced matrix norms $\|A\|_1$ and $\|A\|_\infty$ are treated separately in Theorem 1.2. This is reasonable because, for example, the nilpotent 2-by-2 matrix

$$A := \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \quad a > 0 \tag{6}$$

fulfills the Daugavet equations $1 + a = \|I + A\|_1 = 1 + \|A\|_1 = 1 + \|A\|_\infty = \|I + A\|_\infty$, but $\|A\|_1 = a = \|A\|_\infty$ is not an eigenvalue of A.

Likewise, $r(A) = x^T A x = a/2$ and $r(I + A) = x^T(I + A)x = 1 + a/2$ holds true for $x := 2^{-1/2}(1, 1)^T$, i.e., the Daugavet equation $r(I + A) = 1 + r(A)$ is valid for the numerical radius although $r(A)$ is not an eigenvalue of A. Thus, the assumption $k \geq 1$ in part 3) is necessary.

For $a := 1/2$ and the Frobenius norm $\|\cdot\|_F$, the Daugavet equation $\|I + A\|_F = 3/2 = 1 + \|A\|_F$ holds true again without $\|A\|_F = 1/2$ being an eigenvalue of A. However, $f(0)(\|I\| - 1) = 0$ is not fulfilled because $\|I\|_F = \sqrt{2}$ and $f(0) = 1$ for $f(x) := 1 + x$. Hence, this is not a counterexample to part 2).

Another example is $f(x) := \sum_{k=1}^{\infty} x^{2k}$ and $A := -1/2$ satisfying $f(|A|) = 1/3 = |f(A)|$ even though $|A| = 1/2$ is not an eigenvalue of A. Thus, the condition $c_k c_{k+1} \neq 0$

[1] A vector norm $\|\cdot\|$ on $\mathbb{C}^n$ is *uniformly convex* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - y\| < \epsilon$ for all $x, y \in \mathbb{C}^n$ with $\|x\| = \|y\| = 1$ and $\|\frac{1}{2}(x + y)\| > 1 - \delta$.

[2] A norm $\|\cdot\|$ on M_n is *unitarily invariant* if $\|U^*AV\| = \|A\|$ for all $A \in M_n$ and all unitary $U, V \in M_n$.

for at least some k seems reasonable.

Note that, concerning unitarily invariant norms in part 2) of Theorem 1.2, our Conjecture 1.1 includes weakly unitarily invariant matrix norms as well. However, the numerical radius, which is only weakly unitarily invariant and not a matrix norm, is treated separately in part 3).

In the next section we prove some auxiliary lemmas leading to the reduction of power series to Neumann series. It follows the proof of Theorem 1.2, and the note is finished by some concluding remarks.

2. Reduction of power series to Neumann series

The following lemma is a reformulation of Lemma 2.1 in [5] for finite sums. For the readers convenience a proof is given.[3]

Lemma 2.1. *Let $\|\cdot\|$ be a norm on an $\mathbb{R}$-vector space V and $w_1, \ldots, w_m \in V$, $m \geq 2$. The following statements are equivalent:*

i) $\|\sum_{k=1}^{m} \mu_k w_k\| = \sum_{k=1}^{m} \mu_k \|w_k\|$ *for all $\mu_1, \ldots, \mu_m \in \mathbb{R}_{\geq 0}$*

ii) $\|\sum_{k=1}^{m} w_k\| = \sum_{k=1}^{m} \|w_k\|$

Proof: i) $\Rightarrow$ ii). Take $\mu_k := 1$ for $k = 1, \ldots, m$.
ii) $\Rightarrow$ i). Let $\mu_1, \ldots, \mu_m \in \mathbb{R}_{\geq 0}$. By simultaneous reordering of the μ_k and w_k we may assume without loss of generality that $\mu_1 = \max_{1 \leq k \leq m} \mu_k$. Then, using the triangle inequality, $\mu_1 - \mu_k \geq 0$ for $k = 2, \ldots, m$, and ii), we derive

$$\begin{aligned} \|\sum_{k=1}^{m} \mu_k w_k\| &= \|\mu_1 \sum_{k=1}^{m} w_k - \sum_{k=2}^{m} (\mu_1 - \mu_k) w_k\| \geq \mu_1 \|\sum_{k=1}^{m} w_k\| - \sum_{k=2}^{m} (\mu_1 - \mu_k)\|w_k\| \\ &= \mu_1 \sum_{k=1}^{m} \|w_k\| - \sum_{k=2}^{m} (\mu_1 - \mu_k)\|w_k\| = \sum_{k=1}^{m} \mu_k \|w_k\| \geq \|\sum_{k=1}^{m} \mu_k w_k\| \ , \end{aligned}$$

so that $\|\sum_{k=1}^{m} \mu_k w_k\| = \sum_{k=1}^{m} \mu_k \|w_k\|$. □

Lemma 2.2. *Let V be a normed $\mathbb{R}$-algebra with unit element $\mathbf{1}$ and norm $\|\cdot\|$, and let $f(x) := \sum_{k=0}^{\infty} c_k x^k$ with all $c_k \geq 0$ and $f(0)(\|\mathbf{1}\| - 1) = 0$. Suppose that $f(v) \in V$ and $f(\|v\|) \in \mathbb{R}$ are well-defined for a fixed $v \in V \backslash \{0\}$. Then the following is true:*

i) *If $\|\cdot\|$ is strongly stable, then $\|f(v)\| \leq f(\|v\|)$.*

ii) *If $\|\cdot\|$ is submultiplicative and $h(x) := x^m f(x)$ for some $m \geq 0$ fulfills $\|h(v)\| = h(\|v\|)$, then also $\|f(v)\| = f(\|v\|)$ holds true.*

Proof: i) From $0 = f(0)(\|\mathbf{1}\| - 1) = c_0(\|v^0\| - 1)$ it follows that $c_0\|v^0\| = c_0 = c_0\|v\|^0$. Strong stability of the norm implies $\|v^k\| \leq \|v\|^k$ for $k \geq 1$. Therefore,

$$\|f(v)\| = \|\sum_{k=0}^{\infty} c_k v^k\| \leq \sum_{k=0}^{\infty} c_k \|v\|^k = f(\|v\|).$$

[3] As has been noted by the referee, the lemma extends to infinite sums as well, but this is not needed here.

ii) By i) we have $\|f(v)\| \le f(\|v\|)$, so that submultiplicativity of the norm gives

$$h(\|v\|) = \|h(v)\| = \|v^m f(v)\| \le \|v\|^m \|f(v)\| \le \|v\|^m f(\|v\|) = h(\|v\|).$$

Thus, equality holds and division by $\|v\|^m \neq 0$ proves $\|f(v)\| = f(\|v\|)$. □

Lemma 2.3. *Let V be a normed $\mathbb{R}$-algebra with unit element $\mathbf{1}$ and strongly stable norm $\|\cdot\|$ such that $(V, \|\cdot\|)$ is complete, i.e., a Banach space. Let $f(x) = \sum_{k=0}^{\infty} c_k x^k$ be a power series with nonnegative coefficients $c_k \ge 0$, radius of convergence R, and $f(0)(\|\mathbf{1}\| - 1) = 0$. For $M \subseteq \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ define $f_M(x) := \sum_{k \in M} c_k x^k$. Then, for $v \in V \backslash \{0\}$ with $\|v\| < R$ and $\|f(v)\| = f(\|v\|)$, the following is true:*

i) $\|f_M(v)\| = f_M(\|v\|)$ *for all* $M \subseteq \mathbb{N}_0$

ii) $\|v^k\| = \|v\|^k$ *for all* $k \ge 0$ *with* $c_k \neq 0$

iii) *for a power series* $\tilde{f}(x) = \sum_{k=0}^{\infty} \tilde{c}_k x^k$ *with nonnegative coefficients* $\tilde{c}_k \ge 0$ *such that* $\tilde{c}_k \neq 0$ *implies* $c_k \neq 0$, *radius of convergence* $\tilde{R} > \|v\|$, *and* $\tilde{f}(0)(\|\mathbf{1}\| - 1) = 0$ *we have* $\|\tilde{f}(v)\| = \tilde{f}(\|v\|)$

Proof: i) Let $M \subseteq \mathbb{N}_0$ and set $N := \mathbb{N}_0 \backslash M$. By Lemma 2.2 i) applied to $f_L(x)$ we have $\|f_L(v)\| \le f_L(\|v\|)$ for $L \in \{M, N\}$, so that

$$\begin{aligned} f_M(\|v\|) + f_N(\|v\|) &= f(\|v\|) = \|f(v)\| = \|f_M(v) + f_N(v)\| \le \|f_M(v)\| + f_N(\|v\|) \\ &\le f_M(\|v\|) + f_N(\|v\|). \end{aligned}$$

Subtracting $f_N(\|v\|)$ gives $\|f_M(v)\| = f_M(\|v\|)$.

ii) Taking $M := \{k\}$ in i) yields $c_k \|v^k\| = \|f_M(v)\| = f_M(\|v\|) = c_k \|v\|^k$.

iii) For $m \in \mathbb{N}$ and $k \in M := \{0, \ldots, m\}$ define

$$w_k := c_k v^k \quad \text{and} \quad \mu_k := \begin{cases} \tilde{c}_k c_k^{-1} & \text{if } \tilde{c}_k \neq 0,^4 \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$\mu_k w_k = \tilde{c}_k v^k \ . \tag{7}$$

Using i) and ii) yields

$$\|\sum_{k=0}^{m} w_k\| \;=\; \|f_M(v)\| = f_M(\|v\|) = \sum_{k=0}^{m} \|w_k\| \ .$$

Therefore, defining $\tilde{f}_M$ similar to f_M, Lemma 2.1, (7), and ii) give

$$\|\tilde{f}_M(v)\| \;=\; \|\sum_{k=0}^{m} \mu_k w_k\| = \sum_{k=0}^{m} \mu_k \|w_k\| = \tilde{f}_M(\|v\|) \ .$$

[4] By assumption, $\tilde{c}_k \neq 0$ implies $c_k \neq 0$, so that μ_k is well-defined.

Abbreviating $f_m := f_M$, we conclude

$$\|\tilde{f}(v)\| = \lim_{m\to\infty} \|\tilde{f}_m(v)\| = \lim_{m\to\infty} \tilde{f}_m(\|v\|) = \tilde{f}(\|v\|).$$

□

These preparations allow to state the announced reduction of a general power series with nonnegative coefficients to the (possibly truncated) Neumann series.

Corollary 2.4. *Let V, $\|\cdot\|$, $\mathbf{1}$ be as in Lemma 2.3. Define*

$$g(x) := \delta + \sum_{k=1}^{\infty} x^k$$

where

$$\delta := \begin{cases} 1 & \text{if } \|\mathbf{1}\| = 1 \text{ and } \|\cdot\| \text{ is not submultiplicative,} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Then, for given $v \in V\backslash\{0\}$ with $\|v\| < 1$, the following statements are equivalent:

i) $\|f(v)\| = f(\|v\|)$ *for all* $f(x) = \sum\limits_{k=0}^{\infty} c_k x^k$ *with all* $c_k \geq 0$*, radius of convergence* $R > \|v\|$*, and* $f(0)(\|\mathbf{1}\| - 1) = 0$

ii) $\|f(v)\| = f(\|v\|)$ *for some* $f(x) = \sum\limits_{k=0}^{\infty} c_k x^k$ *with* $c_0 \geq 0$*,* $c_k > 0$ *for* $k \geq 1$*, radius of convergence* $R > \|v\|$*,* $f(0)(\|\mathbf{1}\| - 1) = 0$*, and* $\operatorname{sign}(f(0)) \geq \delta$

iii) $\|g(v)\| = g(\|v\|)$

Proof: i) ⇒ ii). Take $f(x) := g(x)$, note that $g(x)$ has radius of convergence $1 > \|v\|$ and that $g(0)(\|\mathbf{1}\| - 1) = \delta(\|\mathbf{1}\| - 1) = 0$ by (8).

ii) ⇒ iii) follows by Lemma 2.3 iii) with $\tilde{f}(x) := g(x)$.

iii) ⇒ i). Let $f(x) = \sum_{k=0}^{\infty} c_k x^k$ be given with all $c_k \geq 0$, radius of convergence $R > \|v\|$, and $c_0(\|\mathbf{1}\| - 1) = 0$.

Case 1: $c_0 = 0$ or $\delta = 1$. Lemma 2.3 iii) with $(f, \tilde{f}) := (g, f)$ yields $\|f(v)\| = f(\|v\|)$.

Case 2: $c_0 \neq 0 = \delta$. Since $c_0(\|\mathbf{1}\| - 1) = 0$ we have $\|\mathbf{1}\| = 1$, and $\delta = 0$ implies that $\|\cdot\|$ is submultiplicative by (8). Define $h(x) := xf(x)$, so that $h(0) = 0$. Case 1 applied to $h(x)$ gives $\|h(v)\| = h(\|v\|)$, wherefore $\|f(v)\| = f(\|v\|)$ by Lemma 2.2 ii). □

3. Proof of Theorem 1.2

For any strongly stable norm $\|\cdot\|$, Remark 1 proves the easy direction

$$\|A\| \in \operatorname{spec}(A) \;\Rightarrow\; \|f(A)\| = f(\|A\|).$$

For the opposite assume that $\|f(A)\| = f(\|A\|)$ holds true.

First, we consider the cases 1), 2), 3) of Theorem 1.2 and carry out a uniform reduction. The assumptions assure that $c_k c_{k+1} \neq 0$ for some $k \geq 0$, but the proofs are presented under the weaker assumption $c_k c_{k+j} \neq 0$ for some $j \geq 1$. This will prove that $\omega\|A\| \in \operatorname{spec}(A)$ for a j-th root of unity ω, and taking $j := 1$ yields the assertion. By Lemma 2.3 i) with $M := \{k, k+j\}$ and $f_M(x) = c_k x^k + c_{k+j} x^{k+j}$, we have $\|f_M(A)\| = f_M(\|A\|)$. Lemma 2.3 iii) applied to $f_M(x)$ and $h(x) := x^k + x^{k+j}$ gives $\|h(A)\| = h(\|A\|)$. Replacing $h(x)$ by $h(x/\|A\|)$, we also have $\|h(A/\|A\|)\| = h(\|\,A/\|A\|\,\|) = h(1)$, i.e., without loss of generality we may assume that $\|A\| = 1$. However, this is only used in case 3).

Case 1): Set $g(x) := 1 + x^j$, so that $h(x) = x^k g(x)$. By Lemma 2.2 ii) and Lemma 2.3 ii)

$$\|I + A^j\| = \|g(A)\| = g(\|A\|) = 1 + \|A\|^j = 1 + \|A^j\|.$$

Thus, A^j fulfills the Daugavet equation. Theorem 1.3 implies $\|A\|^j = \|A^j\| \in \operatorname{spec}(A^j)$, wherefore $\omega\|A\| \in \operatorname{spec}(A)$ for some j-th root of unity ω. Taking $j := 1$ as noted above gives the assertion.

Case 2): First, we note that a strongly stable, unitarily invariant norm is already submultiplicative, i.e., a matrix norm (cf. [7], p. 211, problem 3). Moreover, a unitarily invariant norm fulfills $\|I\| = 1$ if, and only if $\|\cdot\| = \|\cdot\|_2$ is the Euclidean norm. Since the Euclidean norm is already covered by case 1), we may assume without loss of generality that $\|I\| > 1$. Therefore, the assumption $f(0)(\|I\| - 1) = 0$ implies $c_0 = f(0) = 0$, so that $k \geq 1$ and $h(x) = x^{k-1} g(x)$ with $g(x) := x + x^{j+1}$ is well-defined. By Lemmas 2.3 ii) and 2.2 ii) we have

$$\begin{aligned} \|A^{j+1}\| &= \|A\|^{j+1} , && (9)\\ \|(I + A^j)A\| &= \|g(A)\| = g(\|A\|) = \|A\| + \|A^{j+1}\| = (1 + \|A\|^j)\|A\| . && (10) \end{aligned}$$

The following basic inequalities for unitarily invariant matrix norms will be used:[5]

$$\|XY\| \leq \|X\|_2 \|Y\| \quad \text{and} \quad \|X\|_2 \leq \|X\| \quad \text{for all } X, Y \in M_n. \qquad (11)$$

Taking $X := A^j$ and $Y := A$ in (11), it follows from (9) that

$$\|A^j\|_2 \|A\| \leq \|A^j\|\ \|A\| \leq \|A\|^j \|A\| = \|A\|^{j+1} = \|A^{j+1}\| \leq \|A^j\|_2 \|A\| .$$

Hence, division by $\|A\|$ gives

$$\|A^j\|_2 = \|A^j\| = \|A\|^j. \qquad (12)$$

By (12), (10), and (11) with $X := I + A^j$ and $Y := A$ we get

$$\begin{aligned} \|I + A^j\|_2 \|A\| &\leq (1 + \|A^j\|_2)\|A\| = (1 + \|A^j\|)\|A\| = (1 + \|A\|^j)\|A\| \\ &= \|(I + A^j)A\| \leq \|I + A^j\|_2 \|A\| . \end{aligned}$$

Division by $\|A\|$ gives $\|I + A^j\|_2 = 1 + \|A^j\|_2$, i.e., A^j fulfills the Daugavet equation with respect to the Euclidean norm. Like in case 1), Theorem 1.3 implies $\|A\|^j = \|A^j\| = \|A^j\|_2 \in \operatorname{spec}(A^j)$, so that $\omega\|A\| \in \operatorname{spec}(A)$ for some j-th root of unity ω, and $j := 1$

[5] See, for example, [7], Corollary 3.5.10 on page 206 and problem 3 on page 211.

finishes this case.

Case 3): As noted above, we may assume without loss of generality that $r(A) = 1$. Lemma 2.3 ii) gives $r(A^k) = r(A)^k = 1 = r(A)^{k+j} = r(A^{k+j})$, so that Lemma 2.3 i) yields

$$r(A^k + A^{k+j}) = r(h(A)) = h(r(A)) = r(A)^k + r(A)^{k+j} = r(A^k) + r(A^{k+j}) \ . \tag{13}$$

Let $x \in \mathbb{C}^n$ with $x^*x = 1$ and $|x^*(A^k+A^{k+j})x| = r(A^k+A^{k+j})$. Using (13) we compute

$$r(A^k) + r(A^{k+j}) \quad = \quad |x^*A^kx + x^*A^{k+j}x| \leq |x^*A^kx| + |x^*A^{k+j}x| \leq r(A^k) + r(A^{k+j}) \ ,$$

so that both inequalities become equalities. This implies

$$|x^*A^kx| = r(A^k) = 1 = r(A^{k+j}) = |x^*A^{k+j}x| \quad \text{and} \quad x^*A^kx = x^*A^{k+j}x \ . \tag{14}$$

We will use Ando's characterization [2] of matrices with numerical radius at most 1 as presented in [11] (see also [7], problems 30. and 31. on page 46). The next lemma is an adaption of [11, Theorem 2.1] to our purpose. The proof is deferred to the appendix. A matrix $Z \in M_n$ is called contraction if $\|Z\|_2 \leq 1$, the set of Hermitian n-by-n matrices is denoted by H_n, and $G \succeq 0$ means that $G \in H_n$ is positive semidefinite.

Lemma 3.1. *Let $B \in M_n$ such that $|x^*Bx| = r(B) = 1$ for some $x \in \mathbb{C}^n$ with $x^*x = 1$. Then, there is a contraction $Z \in H_n$ such that $\begin{pmatrix} I+Z & B \\ B^* & I-Z \end{pmatrix} \succeq 0$. All such Z fulfill $\begin{pmatrix} I+Z & B^i \\ (B^i)^* & I-Z \end{pmatrix} \succeq 0$ for all $i \in \mathbb{N}$, and $\begin{pmatrix} I+Z & B \\ B^* & I-Z \end{pmatrix} \begin{pmatrix} x \\ -\beta x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $\beta := \overline{x^*Bx}$.*

Lemma 3.1 applied to $B := A$ supplies a contraction $Z \in H_n$ such that

$$\begin{pmatrix} I+Z & A^k \\ (A^k)^* & I-Z \end{pmatrix} \succeq 0 \quad \text{and} \quad \begin{pmatrix} I+Z & A^{k+j} \\ (A^{k+j})^* & I-Z \end{pmatrix} \succeq 0 \ .$$

For this common Z and the common x from (14), Lemma 3.1 applied to $B := A^k$ and $B := A^{k+j}$, respectively, gives

$$(I+Z)x - \eta A^kx = 0 = (I+Z)x - \zeta A^{k+j}x \ ,$$

where $\eta := \overline{x^*A^kx} = \overline{x^*A^{k+j}x} =: \zeta$ by (14). Thus $v := A^kx$ fulfills $A^jv = v$. Note that $v \neq 0$ because $|x^*v| = |x^*A^kx| = r(A^k) = 1$. As in the previous cases, this means $Av = \omega v$ for a j-th root of unity ω, and $j := 1$ gives the assertion.

Case 4): W.l.o.g. we may assume $\|\cdot\| = \|\cdot\|_\infty$.[6] We first show that we may also assume

$$\|A\|_\infty < 1 \quad \text{and} \quad f(x) = \sum_{k=1}^{\infty} x^k. \tag{15}$$

This is seen as follows. By assumption, there is a $k_0 \geq 0$ such that $c_k \neq 0$ for $k \geq k_0$.

[6]The case $\|\cdot\| = \|\cdot\|_1$ is reduced to the case $\|\cdot\| = \|\cdot\|_\infty$ by taking the transposed matrix A^T.

Lemma 2.3 i) applied to $M := \mathbb{N}_{\geq k_0} = \{k_0, k_0+1, \ldots\}$ gives $\|f_M(A)\|_\infty = f_M(\|A\|_\infty)$. Since $f_M(x) = x^{k_0}h(x)$ for $f(x) = \sum_{k=k_0}^{\infty} c_k x^{k-k_0}$, Lemma 2.2 ii) gives $\|h(A)\|_\infty = h(\|A\|_\infty)$, and we may therefore assume w.l.o.g. $f = h$, i.e., $c_k \neq 0$ for all $k \geq 0$.[7] Defining $\tilde{f}(x) = f(x/d)$ for some $d > \|A\|_\infty$ and $\tilde{A} := A/d$ yields $\|\tilde{A}\|_\infty < 1$, $\tilde{f}(A) = f(\tilde{A})$, and $\|f(\tilde{A})\|_\infty = \|\tilde{f}(A)\|_\infty = \tilde{f}(\|A\|_\infty) = f(\|\tilde{A}\|_\infty)$ by Lemma 2.3 iii). Thus, we may assume w.l.o.g. that $\|A\|_\infty < 1$. Defining $g(x) := \sum_{k=1}^{\infty} x^k$, Corollary 2.4 ii) $\Rightarrow$ iii) shows that $\|g(A)\| = g(\|A\|)$, i.e., we may assume w.l.o.g. that $f(x) = g(x)$ as stated in (15). By Lemma 2.3 ii)

$$\|A^k\|_\infty = \|A\|_\infty^k \quad \text{for all } k \geq 1,$$

so that

$$\rho(A) = \lim_{k\to\infty} \|A^k\|_\infty^{1/k} = \|A\|_\infty \ .$$

Denoting $|A| := (|A_{ij}|) \in M_n$ and using also (5), this implies

$$\rho(A) = \|A\|_\infty = \| \ |A| \ \|_\infty \geq \rho(|A|) \geq \rho(A) \ ,$$

wherefore

$$\|A\|_\infty = \rho(A) = \rho(|A|). \tag{16}$$

First, the case of irreducible A is considered. Then, Perron-Frobenius theory, cf. [6, Theorem 8.4.5], yields

$$A = \alpha D|A|D^{-1}, \quad \alpha \in \mathbb{C},\ |\alpha| = 1,\ \alpha\rho(A) \in \operatorname{spec}(A),\ D \in M_n,\ |D| = I. \tag{17}$$

Let $q \geq 1$ be the cyclicity index of $|A|$. Then

$$e^{2\pi ij/q}\rho(|A|), \quad j = 0, \ldots, q-1, \tag{18}$$

are the eigenvalues of $|A|$ of maximum modulus and

$$e^{2\pi ij/q}\operatorname{spec}(|A|) = \operatorname{spec}(|A|), \quad j = 0, \ldots, q-1, \tag{19}$$

i.e., the spectrum of $|A|$ is invariant under rotations by q-th roots of unity.[8] By Perron-Frobenius theory, cf. [12, Theorem 2.19], there are a permutation matrix $P \in M_n$ and q primitive square matrices $C_1, \ldots, C_q$ such that

$$P|A|^q P^T = \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_q \end{pmatrix} . \tag{20}$$

Hence there is an $m \in \mathbb{N}$ such that

$$C_i^\ell > 0 \quad \text{for all } \ell \geq m \text{ and all } i = 1, \ldots, q,$$

[7] Here submultiplicativity of $\|\cdot\|_\infty$ and $\|I\|_\infty = 1$ is used.

[8] In (18), (19), $i := \sqrt{-1}$ is the complex unit and $e^{2\pi ij/q}$, $j = 0, \ldots, q-1$, are the q-th roots of unity.

where $C_i^\ell > 0$ is meant componentwise. In particular, by (20), this means

$$(|A|^{q\ell})_{ii} > 0 \quad \text{for all } \ell \geq m \text{ and all } i = 1, \ldots, n. \tag{21}$$

Choose a row index $i \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} |f(A)_{ij}| = \|f(A)\|_\infty$. Then, we have

$$\begin{aligned} f(\|A\|_\infty) = \|f(A)\|_\infty &= \sum_{j=1}^{n} \Big| \sum_{k=1}^{\infty} A^k \Big|_{ij} \leq \sum_{j=1}^{n} \sum_{k=1}^{\infty} (|A|^k)_{ij} = \sum_{k=1}^{\infty} \sum_{j=1}^{n} (|A|^k)_{ij} \\ &\leq \sum_{k=1}^{\infty} \|A\|_\infty^k = f(\|A\|_\infty) \ . \end{aligned}$$

Since $| \sum_{k=1}^{\infty} A^k|_{ij} \leq \sum_{k=1}^{\infty} (|A|^k)_{ij}$ for $j = 1, \ldots, n$, this implies

$$\Big| \sum_{k=1}^{\infty} A^k \Big|_{ij} = \sum_{k=1}^{\infty} (|A|^k)_{ij} \quad \text{for } j = 1, \ldots, n.$$

For $j := i$ and distinct $r, s \in \mathbb{N}$ we derive

$$\sum_{k=1}^{\infty} (|A|^k)_{ii} = \Big| \sum_{k=1}^{\infty} A^k \Big|_{ii} \leq |(A^r)_{ii} + (A^s)_{ii}| + \sum_{\substack{k \geq 1 \\ k \neq r,s}} (|A|^k)_{ii} \ .$$

Subtracting $\sum_{\substack{k \geq 1 \\ k \neq r,s}} (|A|^k)_{ii}$ on both sides gives

$$(|A|^r)_{ii} + (|A|^s)_{ii} = |(A^r)_{ii} + (A^s)_{ii}| \ . \tag{22}$$

Taking $r := qm$ and $s := q(m+1)$, (22) and (17) supply

$$\begin{aligned} (|A|^{qm})_{ii} + (|A|^{q(m+1)})_{ii} &= \left| \alpha^{qm} (|A|^{qm})_{ii} + \alpha^{q(m+1)} (|A|^{q(m+1)})_{ii} \right| \\ &= \left| (|A|^{qm})_{ii} + \alpha^q (|A|^{q(m+1)})_{ii} \right| \ . \end{aligned} \tag{23}$$

By (21), $(|A|^{qm})_{ii}, (|A|^{q(m+1)})_{ii} > 0$, so that (23) necessitates $\alpha^q = 1$, i.e., α is a q-th root of unity. From (16), (19), and (17) we finally obtain

$$\|A\|_\infty = \rho(A) = \rho(|A|) \in \text{spec}(|A|) = \alpha \text{spec}(|A|) = \text{spec}(A),$$

i.e., $\|A\|_\infty$ is an eigenvalue of A. This finishes the proof for the irreducible case.

For the reducible case, which is treated now, we need, unexpectedly, quite some effort. Consider the index set

$$J := \{i \in [n] \mid \sum_{j=1}^{n} |f(A)|_{ij} = \|f(A)\|_\infty = f(\|A\|_\infty)\}$$

and define $\tilde{A} := A[J] \in M_m$, $m := |J|$, to be the principal submatrix of A corresponding to J. Then, by the rather technical Lemma 5.2, which is deferred to the appendix, there is a permutation matrix $P \in M_n$ such that

$$P^T A P = \left(\begin{array}{c|c} \tilde{A} & 0 \\ \hline * & * \end{array} \right) .$$

For all $i \in \{1, \ldots, m\}$ this implies

$$\begin{aligned} \sum_{j=1}^{m} |\tilde{A}|_{ij} &= \|\tilde{A}\|_\infty = \|A\|_\infty \quad \text{and} \\ \sum_{j=1}^{m} |f(\tilde{A})|_{ij} &= \|f(\tilde{A})\|_\infty = \|f(A)\|_\infty = f(\|A\|_\infty) = f(\|\tilde{A}\|_\infty). \end{aligned} \tag{24}$$

Like for A, (24) yields $\rho(\tilde{A}) = \|\tilde{A}\|_\infty$ (cf. 16). Choose an irreducible principle subblock B of the reducible normal form of $\tilde{A}$ for which $\rho(B) = \rho(\tilde{A})$ holds true. From

$$\|B\|_\infty \geq \rho(B) = \rho(\tilde{A}) = \|\tilde{A}\|_\infty \geq \|B\|_\infty$$

it follows that $\|B\|_\infty = \rho(B) = \|\tilde{A}\|_\infty$. Perron-Frobenius theory, cf. [12, Lemma 2.8], implies that each row sum of $|B|$ equals $\|B\|_\infty = \|\tilde{A}\|_\infty$, wherefore B must be an isolated block of the reducible normal form of $\tilde{A}$ with eigenvalue $\|B\|_\infty$. By (24) this gives $\|f(B)\|_\infty = f(\|B\|_\infty)$, and it follows from the irreducible case that

$$\|A\|_\infty = \|\tilde{A}\|_\infty = \|B\|_\infty \in \operatorname{spec}(B) \subseteq \operatorname{spec}\tilde{A} \subseteq \operatorname{spec}(A) .$$

This finishes the proof of Theorem 1.2. □

4. Concluding remarks

The Daugavet equation $\|Id + T\| = 1 + \|T\|$ is an intensively investigated subject in functional analysis, see [4], [1], [8], [9], [13], [10], [3] and the references therein. In that context, T is usually an operator in some normed function space in which the identity operator Id has norm 1.

Theorem 1.3 says that for matrix norms induced by uniformly convex vector norms, like the ℓ^p-norms for $1 < p < \infty$, the validity of the Daugavet equation $\|I + A\| = 1 + \|A\|$ forces $\|A\|$ to be an eigenvalue of $A \in M_n$. As well-known and recalled by examples in the introduction this is not true for other norms on M_n like the ℓ^1- and the ℓ^∞-norm, many unitarily invariant norms like the Frobenius norm, or for the numerical radius. Therefore, one may ask for alternatives to replace the Daugavet equation in Theorem 1.3 for these norms, still confirming that $\|A\|$ an eigenvalue of A. In this light our main Theorem 1.2 says that

$$\|A + A^2\| = \|A\| + \|A\|^2 \tag{25}$$

is the appropriate analogue for unitarily invariant matrix norms and for the numerical radius. Likewise the (scaled) Neumann series

$$\|\sum_{k=0}^{\infty}(A/d)^k\| = \sum_{k=0}^{\infty}\|A/d\|^k \qquad \text{for some } d > \|A\| \tag{26}$$

are suitable analogues for the ℓ^1- and the ℓ^∞-norm.

Note that (25) implies the Daugavet equation if $\|\cdot\|$ is submultiplicative (see Lemma 2.2 ii)), so that (25) is a stronger condition than the Daugavet equation in this case. We are not aware of any investigation in equation (25) in the literature around the Daugavet equation.

Next, we mention that Conjecture 1.1 includes all matrix norms induced by vector norms, and that one may wonder why this important basic class of induced norms does not occur as such in our main Theorem 1.2. A proof must cover in particular induced ℓ^p-norms for $1 < p < \infty$ on the one hand and for $p \in \{1, \infty\}$ on the other. We could not find a unified proof but had to treat those cases separately using individual arguments.

Finally, one may ask what Theorem 1.2 tells us for specific power series with non-negative coefficients like, for example, the exponential function $f(x) := \exp(x)$. Since $\exp(0) = 1 \neq 0$, the assumption $f(0)(\|I\| - 1) = 0$ of Theorem 1.2 is not fulfilled for unitarily invariant matrix norms different from the Euclidean norm, for example, the Frobenius norm $\|\cdot\|_F$. Although all Taylor coefficients of $\exp(x)$ are positive, the result is not true without this assumption as by the nilpotent 2-by-2 matrix A defined in (6) with $a \approx 0.3817$ being the root of $\exp(z) - \sqrt{2 + z^2}$. Then

$$\|\exp(A)\|_F = \|I + A\|_F = \sqrt{2 + a^2} = \exp(a) = \exp(\|A\|_F) \ ,$$

but $\|A\|_F = a$ is not an eigenvalue of A.

On the other hand, the assumption $f(0)(\|I\| - 1) = 0$ is satisfied for the numerical radius as well as for all ℓ_p-norms, $1 \leq p \leq \infty$, because $r(I) = 1 = \|I\|_p$. As a consequence, $r(\exp(A)) = \exp(r(A))$ holds true if, and only if, $r(A)$ is an eigenvalue of A, and $\|\exp(A)\|_p = \exp(\|A\|_p)$ if, and only if, $\|A\|_p$ is an eigenvalue of A.

5. Appendix

5.1. Proof of Lemma 3.1

By [11, Theorem 2.1] a) there is a contraction $Z \in H_n$ such that $M := \begin{pmatrix} I+Z & B \\ B^* & I-Z \end{pmatrix} \succeq 0$. For $v := \begin{pmatrix} x \\ -\beta x \end{pmatrix}$ we have $v^*Mv = 2 - 2\mathrm{Re}(\beta x^* Bx) = 0$, so that $M \succeq 0$ implies $Mv = 0$. It remains to show that

$$\begin{pmatrix} I+Z & B^i \\ (B^i)^* & I-Z \end{pmatrix} \succeq 0 \quad \text{for all } i \in \mathbb{N}. \tag{27}$$

This is done by repeating arguments given after the proof of [11, Theorem 2.1]. By [6, Theorem 7.7.11] the following equivalence holds true for positive semidefinite $A, C \in H_n$ (and arbitrary $B \in M_n$):

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \succeq 0 \quad \Leftrightarrow \quad \text{There is a contraction } X \in M_n \text{ such that } B = A^{1/2} X C^{1/2}. \tag{28}$$

Since $Z \in H_n$ is a contraction, $A := I + Z$ and $C := I - Z$ are positive semidefinite. Thus (28) supplies a contraction X such that $B = (I+Z)^{1/2} X (I-Z)^{1/2}$. Taking i-th powers yields

$$B^i = (I+Z)^{1/2} X[(I-Z^2)^{1/2} X]^{i-1} (I-Z)^{1/2}.$$

Since Z is a Hermitian contraction, this is also true for $(I - Z^2)^{1/2}$. Therefore, $Y := X[(I-Z^2)^{1/2}X]^{i-1}$ fulfills $\|Y\|_2 \leq \|X\|_2 [\|(I-Z^2)^{1/2}\|_2 \|X\|_2]^{i-1} \leq 1$, i.e., Y is a contraction. Hence $B^i = (I+Z)^{1/2} Y (I-Z)^{1/2}$ and (28) prove (27).

5.2. Auxiliary lemmas for the ℓ^∞-norm

Abbreviate $[n] := \{1, \ldots, n\}$ and let $\|v\|_1 := \sum_{i=1}^n |v_i|$ denote the 1-norm of $v \in \mathbb{C}^n$. For $A \in M_n$ and $i \in [n]$ set $A_{i:} := (A_{i1}, \ldots, A_{in})^T$, so that $\|A\|_\infty := \max_{i \in [n]} \|A_{i:}\|_1$.

We will frequently use that two nonzero complex numbers x and y point in the same direction of the complex plane if, and only if, $x\overline{y} > 0$, where the latter implicitly indicates that $x\overline{y}$ is real. Thus, $|x + y| = |x| + |y|$ holds true if, and only if $x\overline{y} \geq 0$.

Lemma 5.1. *Let $X, Y \in M_n$, and $i \in [n]$ satisfy $\|(XY)_{i:}\|_1 = \|XY\|_\infty = \|X\|_\infty \|Y\|_\infty \neq 0$. Then, for all $j, \ell \in [n]$ we have*

- i) $X_{i\ell} Y_{\ell j} \overline{(XY)_{ij}} \geq 0$ *and the inequality is strict if* $X_{i\ell} Y_{\ell j} \neq 0$,
- ii) $X_{i\ell} = 0$ *or* $\|Y_{\ell :}\|_1 = \|Y\|_\infty$.

Proof: Straight forward computation gives

$$\begin{aligned} \|XY\|_\infty &= \|(XY)_{i:}\|_1 \\ &= \sum_{j=1}^n |(XY)_{ij}| = \sum_{j=1}^n \Big| \sum_{\ell=1}^n X_{i\ell} Y_{\ell j} \Big| \leq \sum_{j=1}^n \sum_{\ell=1}^n |X_{i\ell}| \; |Y_{\ell j}| \qquad (29) \\ &= \sum_{\ell=1}^n |X_{i\ell}| \sum_{j=1}^n |Y_{\ell j}| \leq \sum_{\ell=1}^n |X_{i\ell}| \; \|Y\|_\infty \qquad (30) \\ &\leq \|X\|_\infty \|Y\|_\infty = \|XY\|_\infty. \end{aligned}$$

Thus, equality holds in (29) proving i), and in (30) proving ii). □

Lemma 5.2. *Let $f(x) := \sum_{k=1}^\infty x^k$ and $A \in M_n \backslash \{0\}$ with $\|A\|_\infty < 1$ and $\|f(A)\|_\infty = f(\|A\|_\infty)$. Define $J := \{i \in [n] \mid \|f(A)_{i:}\|_1 = \|f(A)\|_\infty\}$. Then, there is a permutation matrix $P \in M_n$ such that*

$$P^T A P = \left(\begin{array}{c|c} A[J] & 0 \\ \hline * & * \end{array} \right)$$

where $A[J]$ denotes the principal submatrix of A corresponding to the index set J.

Proof: Let $i \in J$ and $j \in [n]$ such that $A_{ij} \neq 0$. The assertion follows if $j \in J$ can be shown. Lemma 2.3 ii) yields $\|A^k\|_\infty = \|A\|_\infty^k$ for all $k \geq 1$. Thus, Lemma 5.1 ii) applied to $X := A$, $Y := A^k$, $\ell := j$, and using $X_{i\ell} = A_{ij} \neq 0$ gives

$$\|A^k_{j:}\|_1 = \|A^k\|_\infty = \|A\|_\infty^k \quad \text{for all } k \in \mathbb{N}. \tag{31}$$

By assumption and definition of J we have

$$\begin{aligned} f(\|A\|_\infty) &= \|f(A)\|_\infty = \|f(A)_{i:}\|_1 = \sum_{\ell=1}^n \Big|\sum_{k=1}^\infty (A^k)_{i\ell}\Big| \leq \sum_{\ell=1}^n \sum_{k=1}^\infty |(A^k)_{i\ell}| \qquad (32) \\ &\leq \sum_{\ell=1}^n \sum_{k=1}^\infty (|A|^k)_{i\ell}| = \|f(|A|)_{i:}\|_1 \leq \|f(|A|)\|_\infty \leq f(\|\,|A|\,\|_\infty) = f(\|A\|_\infty)\ . \end{aligned}$$

Thus, equality holds in (32) which implies $|\sum_{k=1}^\infty (A^k)_{i\ell}| = \sum_{k=1}^\infty |(A^k)_{i\ell}|$ for all $\ell \in [n]$, and therefore

$$(A^k)_{i\ell}\ \overline{(A^{k'})_{i\ell}} \geq 0 \quad \text{for all } \ell \in [n] \text{ and all } k, k' \in \mathbb{N}. \tag{33}$$

In plain English this means that for any fixed $\ell \in [n]$, all nonzero $(A^k)_{i\ell}$, $k \in \mathbb{N}$, point in the same direction in the complex plane. Let $\ell \in [n]$ and $k, k' \in \mathbb{N}$ and suppose that $(A^k)_{j\ell} \neq 0 \neq (A^{k'})_{j\ell}$. Lemma 5.1 i) applied to $X := A$ and $Y := A^k$ gives $A_{ij}\ (A^k)_{j\ell}\ \overline{(A^{k+1})_{i\ell}} > 0$ and likewise $A_{ij}\ (A^{k'})_{j\ell}\ \overline{(A^{k'+1})_{i\ell}} > 0$. Therefore,

$$\begin{aligned} 0 &< A_{ij}\ (A^k)_{j\ell}\ \overline{(A^{k+1})_{i\ell}}\ \ \overline{A_{ij}\ (A^{k'})_{j\ell}\ \overline{(A^{k'+1})_{i\ell}}} \\ &= (A^k)_{j\ell}\ \overline{(A^{k'})_{j\ell}}\ |A_{ij}|^2\ \overline{(A^{k+1})_{i\ell}}\ (A^{k'+1})_{i\ell}\ . \end{aligned}$$

By (33) we have $\overline{(A^{k+1})_{i\ell}}\ (A^{k'+1})_{i\ell} > 0$, so that $(A^k)_{j\ell}\ \overline{(A^{k'})_{j\ell}} > 0$, i.e., we proved

$$(A^k)_{j\ell}\ \overline{(A^{k'})_{j\ell}} \geq 0 \quad \text{for all } \ell \in [n] \text{ and all } k, k' \in \mathbb{N}.$$

This implies

$$\Big|\sum_{k=1}^\infty (A^k)_{j\ell}\Big| = \sum_{k=1}^\infty |(A^k)_{j\ell}| \quad \text{for all } \ell \in [n]. \tag{34}$$

From (34) and (31) we conclude

$$\begin{aligned} \|f(A)_{j:}\|_1 &= \sum_{\ell=1}^n \Big|\sum_{k=1}^\infty (A^k)_{j\ell}\Big| = \sum_{\ell=1}^n \sum_{k=1}^\infty |(A^k)_{j\ell}| = \sum_{k=1}^\infty \sum_{\ell=1}^n |(A^k)_{j\ell}| = \sum_{k=1}^\infty \|(A^k)_{j:}\|_1 \\ &= \sum_{k=1}^\infty \|A^k\|_\infty = \sum_{k=1}^\infty \|A\|_\infty^k = f(\|A\|_\infty) = \|f(A)\|_\infty\ , \end{aligned}$$

which means $j \in J$. □

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