On norms of principal submatrices

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Abstract

Let a norm on the set M_n of real or complex n-by-n matrices be given. We investigate the question of finding the largest constants α_n and β_n such that for each $A \in M_n$ the average of the norms of its (n−1)-by-(n−1) principal submatrices is at least α_n times the norm of A, and such that the maximum of the norms of those principal submatrices is at least β_n times the norm of A.

For a variety of classical norms including induced ℓ^p-norms, weakly unitarily invariant norms, and entrywise norms we give lower and upper bounds for α_n and β_n. In several cases α_n and β_n are explicitly determined.

Key words: matrix norms, principal submatrices, norm inequalities
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1. Notation and Introduction

For $n \in \mathbb{N} := \{1, 2, 3, \dots\}$ let M_n denote the set of real or complex square matrices of order n. The following notation is used where $A = (a_{ij}) \in M_n$:

I	the $n \times n$ identity matrix
E	the $n \times n$ matrix of all 1's
e_i	the i-th column of I
J_k	$I - e_k e_k^T, \ 1 \le k \le n$
A_k	$J_k A J_k,$ the matrix A with k-th row and column set to zero
$A_{\{k\}}$	the principal submatrix by deleting the k-th row and column
$\operatorname{diag}(A)$	the matrix consisting of the diagonal of A
$\sigma_i(A)$	the i-th singular value of A in decreasing order
$r(A)$	$\max\{\lvert x^*Ax\rvert \mid x \in \mathbb{C}^n, x^*x = 1\},$ the numerical radius
$\varrho(A)$	the spectral radius
$\lVert A\rVert_p$	$\max\{\lVert Ax\rVert_p \mid \lVert x\rVert_p = 1\},$ the induced ℓ^p-norm, $1 \le p \le \infty$
$\lVert A\rVert_U$	an arbitrary weakly unitarily invariant matrix norm
$\lVert A\rVert_{(p)}$	$\left(\sum_{i,j} \lvert a_{ij}\rvert^p\right)^{1/p},$ the entrywise p-norm for $1 \le p < \infty$
$\lVert A\rVert_{(\infty)}$	$\max_{i,j} \lvert a_{ij}\rvert,$ the entrywise infinity norm

Let a matrix order $n \ge 2$ and a norm $\|\cdot\|$ on M_n be fixed.[1] This note is about bounding the largest constants α_n and β_n satisfying

$$\frac{1}{n}\sum_{k=1}^{n} \|A_k\| \ge \alpha_n \|A\| \quad \text{and} \quad \max_{1\le k\le n} \|A_k\| \ge \beta_n \|A\| \quad \text{for all } A \in M_n.^2 \tag{1}$$

Some conditions must be imposed on $\|\cdot\|$ to exclude pathological cases.[3] Note that $\alpha_n \le \beta_n$, $\alpha_n \le (n-2)/n$, and $\beta_n \le 1$ for any norm because

[1] We do not require the norm to be submultiplicative. For example, our results also cover the numerical radius $r(A)$. Actually, all our findings are valid for seminorms, i.e., $\|A\| = 0 \ \Rightarrow \ A = 0$ is not needed.

[2] We prefer working with the embeddings $A_k = J_k A J_k$ of the principal submatrices $A_{\{k\}}$ of A into M_n rather than working with the $A_{\{k\}} \in M_{n-1}$ directly because this allows us to consider only one single norm without taking care of dimensional inheritance properties.

[3] $\|A\| := |\sum_{i,j} a_{ij}| + \varepsilon \sum_{i,j} |a_{ij}|$ defines a norm on M_n for any $\varepsilon > 0$. The matrix $A := E - (n-1)I$ fulfills $\|A\| = n + n(2n-3)\varepsilon$ and $\|A_k\| = 2(n-1)(n-2)\varepsilon$ for all $k \in \{1, \dots, n\}$, so that α_n and β_n defined in (1) become arbitrarily small for $\varepsilon \to 0$.

$A := e_1 e_2^T$ fulfills $A_k = A$ for $k \neq 1, 2$ and $A_1 = 0 = A_2$, implying

$$\frac{1}{n}\sum_{k=1}^{n} \|A_k\| = \frac{n-2}{n}\|A\| \qquad \text{and} \qquad \max_{1\le k\le n} \|A_k\| \le \|A\|. \tag{2}$$

Furthermore, $\alpha_n = (n-2)/n$ and $\beta_n = 1$ are simultaneously attained by the entrywise infinity norm (with $n \ge 3$ for β_n).

For many common norms asymptotically sharp bounds for α_n and β_n will be derived. In several cases the exact values of α_n and β_n are explicitly determined. The results are summarized in Table 1 in which the third and fourth column either state exact values or enclosing intervals for α_n and β_n.

Nr.	**Norm**	$\boldsymbol{\alpha_n}$ [4]	$\boldsymbol{\beta_n}$	**Example**[5]
1	$\|\cdot\|_p$, $1 \le p \le \infty$	$\left[\frac{(n-1)(n-2)}{n^2}, \frac{n-2}{n}\right]$	$\left[\frac{n-2}{n}, \frac{n-1}{n}\right]$	E
1.1	$\|\cdot\|_2$, $r(\cdot)$	$\left[\frac{(n-1)(n-2)}{n^2}, \frac{n-2}{n}\right]$	$\left[\frac{n-2}{n}, \frac{n-1}{n+1}\right]$	$E - \frac{n-1}{2} I$
1.2	$\|\cdot\|_1$, $\|\cdot\|_\infty$	$\frac{n-2}{n}$	$\frac{n-2}{n-1}$	$E - I$
2	any weakly unitarily invariant norm	$\left[\frac{(n-1)(n-2)}{n^2}, \frac{n-2}{n}\right]$	$\left[\frac{(n-1)(n-2)}{n^2}, \frac{n-1}{n}\right]$	E
3	$\|\cdot\|_{(p)}$, $1 \le p \le \infty$	$\frac{n-2}{n}$	$\left(\frac{n-2}{n}\right)^{1/p}$	$E - I$

Table 1: Bounds for α_n and β_n as in (1).

In what follows we will prove the statements in Table 1.

2. Hermitian matrices

For Hermitian $A \in M_n$, denote the eigenvalues of A by

$$\lambda_1 \ge \ldots \ge \lambda_n$$

and for $1 \le k \le n$ the eigenvalues of $A_{\{k\}} \in M_{n-1}$ by

$$\lambda_{k,1} \ge \ldots \ge \lambda_{k,n-1}.$$ [6]

[4] Recall that the upper bound $(n-2)/n$ for α_n is attained by $A := e_1 e_2^T$ for all norms.

[5] Example of a matrix realizing the upper bound or the exact value of β_n. Based on numerical evidence we conjecture that $(n-1)/(n+1)$ is the exact value of β_n in Nr. 1.1.

[6] Recall that $A_{\{k\}}$ is the principal submatrix of order $n-1$ of A obtained by deleting the k-th row and column.

Thompson [5] (see also [6]) showed

$$\sum_{k=1}^{n} \frac{\lambda_1 - \lambda_{k,1}}{\lambda_1 - \lambda_n} \le 1$$

provided that $\lambda_1 \neq \lambda_n$. If $\lambda_1 \ge -\lambda_n$, this implies

$$\sum_{k=1}^{n} \lambda_{k,1} \ge \lambda_n + (n-1)\lambda_1 \ge (n-2)\lambda_1 = (n-2)\varrho(A) \ , \tag{3}$$

and if $\lambda_1 < -\lambda_n$ application to $-A$ gives

$$\sum_{k=1}^{n} \lambda_{k,n-1} \le \lambda_1 + (n-1)\lambda_n < (n-2)\lambda_n = -(n-2)\varrho(A) \ .$$

Hence $\varrho(A_{\{k\}}) = \max(\lambda_{k,1} \ , \ -\lambda_{k,n-1})$ implies for Hermitian A

$$\max_{1\le k\le n} \varrho(A_{\{k\}}) \ge \frac{1}{n} \sum_{k=1}^{n} \varrho(A_{\{k\}}) \ge \frac{n-2}{n}\varrho(A) \ , \tag{4}$$

which is, of course, also true for $\lambda_1 = \lambda_n$.[7] The matrix $A := E - \frac{n-1}{2} I$ satisfies

$$\max_{1\le k\le n} \varrho(A_{\{k\}}) = \frac{n-1}{n+1}\varrho(A) \ . \tag{5}$$

Thus, (4) and (5) supply the enclosure for β_n stated in Nr. 1.1 of Table 1 for the spectral norm $\|\cdot\|_2$ and Hermitian A, where $\|A_{\{k\}}\|_2 = \|A_k\|_2$ is used. As a side note we remark that for positive semidefinite A, $\lambda_n \ge 0$ and (3) yield (cf. [3])

$$\max_{1\le k\le n} \varrho(A_{\{k\}}) \ge \frac{1}{n} \sum_{k=1}^{n} \varrho(A_{\{k\}}) \ge \frac{n-1}{n}\varrho(A) \ .$$

This is sharp for $A = E$ where

$$\max_{1\le k\le n} \varrho(A_{\{k\}}) = \frac{1}{n} \sum_{k=1}^{n} \varrho(A_{\{k\}}) = \frac{n-1}{n}\varrho(A) \ .$$

[7] In fact, this well-known result of Thompson motivated this note.

3. General matrices

Define $J_k := I - e_k e_k^T$. Then $A_k := J_k A J_k$ is the matrix obtained by setting the k-th row and column of A to zero. The following splitting is easy to check and crucial for our further considerations:

$$\sum_{k=1}^{n} A_k = (n-2)A + \mathrm{diag}(A) \ . \tag{6}$$

3.1. A weak condition suitable for weakly unitarily invariant norms

Suppose that for all $A = (a_{ij}) \in M_n$ the following condition holds true:

$$\|\mathrm{diag}(A)\| \le \|A\| \ . \tag{7}$$

By (6) and (7) applied to A_k for $1 \le k \le n$, we have

$$\begin{aligned}
(n-2)\|A\| &= \|\sum_{k=1}^{n} A_k - \mathrm{diag}(A)\| = \|\sum_{k=1}^{n}\left(A_k - \frac{1}{n-1}\mathrm{diag}(A_k)\right)\| \\
&\le \sum_{k=1}^{n}\left(\|A_k\| + \frac{1}{n-1}\|\mathrm{diag}(A_k)\|\right) \le \sum_{k=1}^{n}\left(1 + \frac{1}{n-1}\right)\|A_k\| \\
&= \frac{n^2}{n-1}\cdot\frac{1}{n}\sum_{k=1}^{n}\|A_k\| \le \frac{n^2}{n-1}\max_{1\le k\le n}\|A_k\| \ ,
\end{aligned}$$

so that

$$\frac{(n-1)(n-2)}{n^2}\|A\| \le \frac{1}{n}\sum_{k=1}^{n}\|A_k\| \le \max_{1\le k\le n}\|A_k\| \ .$$

Thus, all norms satisfying (7) fulfill

$$\beta_n \ge \alpha_n \ge (n-1)(n-2)/n^2 \ . \tag{8}$$

Indeed, this lower bound for α_n and β_n is realized by the norm

$$\|A\| := \sum_{k=1}^{n} \max\left(|a_{kk}|, \frac{1}{2}\max_{j\ne k}|a_{kj}|, |\sum_{j=1}^{n} a_{kj}|\right) \ , \tag{9}$$

which fulfills (7) and for which $A := E - \frac{n}{2}I$ gives $\|A\| = n^2/2$ and

$$\max_{1\le k\le n}\|A_k\| = \frac{1}{n}\sum_{k=1}^{n}\|A_k\| = \frac{(n-1)(n-2)}{2} = \frac{(n-1)(n-2)}{n^2}\|A\| \ .$$

It is well known that weakly unitarily invariant norms $\|\cdot\|_U$ fulfill (7). This is easily seen as follows (cf. [1]). Let $A = (a_{ij}) \in M_n$ be given, let ω be a primitive n-th root of unity, and consider the unitary diagonal matrix $V := \mathrm{diag}(1, \omega, \omega^2, \ldots, \omega^{n-1})$. Then,

$$\mathrm{diag}(A) = \frac{1}{n} \sum_{k=0}^{n-1} V^{*k} A V^k \ ,$$

wherefore weak unitary invariance of $\|\cdot\|_U$ implies

$$\|\mathrm{diag}(A)\|_U \leq \frac{1}{n} \sum_{k=0}^{n-1} \|V^{*k} A V^k\|_U = \|A\|_U \ .$$

It is easy to see that $\|vv^T\|_U = v^T v \|e_1 e_1^T\|_U$ for each $v \in \mathbb{C}^n$. Therefore, the matrix $A = E \in M_n$ fulfills $\|A\|_U = n\|e_1 e_1^T\|_U$ and $\|A_k\|_U = (n-1)\|e_1 e_1^T\|_U$ for $k \in \{1, \ldots, n\}$, so that

$$\max_{1 \leq k \leq n} \|A_k\|_U = \frac{n-1}{n} \|A\|_U \ . \tag{10}$$

Summarizing, (8), (10), and the first equality in (2) supply

$$\alpha_n \in \left[\frac{(n-1)(n-2)}{n^2}, \frac{n-2}{n}\right] \quad \text{and} \quad \beta_n \in \left[\frac{(n-1)(n-2)}{n^2}, \frac{n-1}{n}\right] \tag{11}$$

for any weakly unitarily invariant norm. This is stated in Nr. 2 of Table 1. However, we could not find an example of a weakly unitarily invariant norm for which α_n attains the lower bound $(n-1)(n-2)/n^2$, and by strong numerical evidence we conjecture that $\alpha_n = (n-2)/n$ for these norms. If true, this is also the exact value of β_n for the Schatten 1-norm; for all $n \geq 3$ the ratio $(n-2)/n$ is realized for the permutation matrix mapping $(1, \ldots, n)$ into $(3, \ldots, n, 1, 2)$.

3.2. A stronger condition suitable for ℓ^p-norms and the numerical radius

Now suppose that, instead of (7), the following stronger condition holds true for all $A = (a_{ij}) \in M_n$, $k \in \{1, \ldots, n\}$, and $c \in \mathbb{C}$:

$$\|A\| \geq \|a_{kk} e_k e_k^T\| \quad \text{and} \quad \|A_k + c e_k e_k^T\| \leq \max\left(\|A_k\|, \|c e_k e_k^T\|\right) . ^{8} \tag{12}$$

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[8] In [4] and [2] the *maximum property* $N(A \oplus B) = \max\left(N(A), N(B)\right)$ of certain induced matrix norms $N(\cdot)$ on direct sums $A \oplus B$ was introduced and characterized; the second part $\|A_k + c e_k e_k^T\| \leq \max\left(\|A_k\|, \|c e_k e_k^T\|\right)$ of (12) is a weakening of that property.

In plain English, first, the norm of a matrix is bounded from below by the norm of $a_{kk}e_ke_k^T$, the matrix having a_{kk} as k-th diagonal entry and zeros elsewhere, and second, replacing the k-th row and column of A by zeros but the k-th diagonal element by c, the norm of the resulting matrix $A_k + ce_ke_k^T$ is bounded from above by the norm of A_k or by the norm of the matrix $ce_ke_k^T$.

Indeed, (12) implies (7) as seen by applying the second inequality in (12) successively to diag(A) and then using the first inequality in (12), so that

$$\|\mathrm{diag}(A)\| \le \max_{1\le k\le n} \|a_{kk}e_ke_k^T\| \le \|A\|.$$

It is straightforward to check that the ℓ^p-norms $\|A\|_p$, $1 \le p \le \infty$, and the numerical radius $r(A)$ fulfill (12). Contrary, the second inequality in (12) is, for example, not satisfied for the Frobenius norm or, more generally, for the Schatten p-norms if $1 \le p < \infty$. Also the Ky Fan k-norms for $k \ge 2$ and the entrywise p-norms $\|A\|_{(p)}$ for $1 \le p < \infty$ do not fulfill (12).

For $A = (a_{ij}) \in M_n$ and $B_{[k]} := A_k - a_{kk}e_ke_k^T$ the splitting (6) transforms into

$$(n-2)A = \sum_{k=1}^{n} A_k - \mathrm{diag}(A) = \sum_{k=1}^{n} B_{[k]}. \tag{13}$$

For each $k \in \{1, \ldots, n\}$, the first inequality in (12) implies $\|a_{kk}e_ke_k^T\| \le \|A_\ell\|$ for all $\ell \ne k$. Thus, again using (12),

$$\|B_{[k]}\| \le \max\left(\|A_k\|, \|a_{kk}e_ke_k^T\|\right) \quad \text{and} \quad \max_{1\le k\le n} \|B_{[k]}\| \le \max_{1\le k\le n} \|A_k\| .$$

By (13), it follows

$$(n-2)\|A\| = \|\sum_{k=1}^{n} B_{[k]}\| \le n \max_{1\le k\le n} \|B_{[k]}\| \le n \max_{1\le k\le n} \|A_k\|$$

and therefore

$$\frac{n-2}{n}\|A\| \le \max_{1\le k\le n} \|A_k\|.$$

Thus, norms satisfying (12) fulfill

$$\beta_n \ge \frac{n-2}{n}. \tag{14}$$

One may ask whether this is also a lower bound for α_n for such norms. However, this is not true as seen by the following norm derived from (9) by replacing the outer sum by a maximum:

$$\|A\| := \max_{1\le k\le n} \left(|a_{kk}|, \frac{1}{2}\max_{j\ne k}|a_{kj}|, \left|\sum_{j=1}^{n} a_{kj}\right| \right) .$$

This norm fulfills (12) and the matrix A with first row $(-(n-2)/2, 1, \ldots, 1)$ and zeros elsewhere fulfills $\|A\| = n/2$, $\|A_1\| = 0$, and $\|A_k\| = (n-2)/2$ for $k \ne 1$, so that

$$\frac{1}{n}\sum_{k=1}^{n}\|A_k\| = \frac{(n-1)(n-2)}{2n} = \frac{(n-1)(n-2)}{n^2}\|A\| .$$

Thus, the lower bound (8) for norms satisfying (7) does not increase for norms satisfying the stronger condition (12).

In conclusion, (8) and (14) prove the lower bounds stated in Nr. 1 and Nr. 1.1 of Table 1. The upper bound for β_n in Nr. 1 is realized by $A := E$ for all $1 \le p \le \infty$. The upper bound for β_n in Nr. 1.1 is realized by the matrix $A := E - \frac{n-1}{2}I$ considered in (5) which is symmetric, so that spectral radius, spectral norm, and numerical radius coincide. Note that condition (12) on its own only implies the upper bound $\beta_n \le 1$ as by the entrywise infinity norm for $n \ge 3$.

The norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ satisfy (12), but sharper estimates are possible. Let $i \in \{1, \ldots, n\}$ be such that $\|A\|_\infty = \sum_{j=1}^{n}|a_{ij}|$. Then,

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n}\|A_k\|_\infty &\ge \frac{1}{n}\sum_{k\ne i}\|A_k\|_\infty \ge \frac{1}{n}\sum_{k\ne i}\sum_{j\ne k}|a_{ij}| = \frac{1}{n}\sum_{k\ne i}(\|A\|_\infty - |a_{ik}|) \\ &= \frac{n-2}{n}\|A\|_\infty + \frac{1}{n}|a_{ii}| \ge \frac{n-2}{n}\|A\|_\infty , \end{aligned}$$

and using also (2) we conclude $\alpha_n = (n-2)/n$.

Furthermore, if $|a_{i\ell}| \le \frac{1}{n-1}\|A\|_\infty$ for some ℓ with $\ell \ne i$, then

$$\|A_\ell\|_\infty \ge \sum_{j\ne \ell}|a_{ij}| = \|A\|_\infty - |a_{i\ell}| \ge \left(1 - \frac{1}{n-1}\right)\|A\|_\infty = \frac{n-2}{n-1}\|A\|_\infty .$$

If $|a_{i\ell}| \ge \frac{1}{n-1}\|A\|_\infty$ for all $\ell \ne i$, then, for all $k \ne i$, we have

$$\|A_k\|_\infty \ge \sum_{\ell\ne i,k}|a_{i\ell}| \ge \frac{n-2}{n-1}\|A\|_\infty .$$

Thus, in either case

$$\max_k \|A_k\|_\infty \geq \frac{n-2}{n-1}\|A\|_\infty$$

and equality holds true for $A = E - I$. This yields $\beta_n = (n-2)/(n-1)$. Transposing gives the same results for $\|\cdot\|_1$. This proves Nr. 1.2 of Table 1.

3.3. Entrywise norms

Finally, we consider the entrywise p-norms $\|A\|_{(p)}$ with $1 \leq p \leq \infty$. For this case we rewrite Equation (6) as follows:

$$\sum_{k=1}^{n} A_k = (n-2)\big(A - \mathrm{diag}(A)\big) + (n-1)\mathrm{diag}(A). \tag{15}$$

Let $p \in [1, \infty)$. The two matrices $D := \mathrm{diag}(A)$ and $A_0 := A - D$ appearing on the right-hand side of (15) fulfill $A_0 \circ D = 0$ and we compute

$$\begin{aligned}
(n-2)^p\|A\|_{(p)}^p &= (n-2)^p(\|A_0\|_{(p)}^p + \|D\|_{(p)}^p) \\
&\leq (n-2)^p\|A_0\|_{(p)}^p + (n-1)^p\|D\|_{(p)}^p \\
&= \|(n-2)A_0 + (n-1)D\|_{(p)}^p \\
&= \|\sum_{k=1}^{n} A_k\|_{(p)}^p \leq n^p\left(\frac{1}{n}\sum_{k=1}^{n}\|A_k\|_{(p)}\right)^p
\end{aligned}$$

implying

$$\frac{n-2}{n}\|A\|_{(p)} \leq \frac{1}{n}\sum_{k=1}^{n}\|A_k\|_{(p)}\ .$$

Using (2) again gives

$$\alpha_n = \frac{n-2}{n}\ . \tag{16}$$

Now, define $\hat{A} := (|a_{ij}|^p)$. Like before, the two matrices $\hat{D} := \mathrm{diag}(\hat{A})$ and $\hat{A}_0 := \hat{A} - \hat{D}$ fulfill $\hat{A}_0 \circ \hat{D} = 0$ and applying (15) to $\hat{A}$ gives

$$\begin{aligned}
(n-2)\|A\|_{(p)}^p &= (n-2)(\|\hat{A}_0\|_{(1)} + \|\hat{D}\|_{(1)}) \\
&\leq (n-2)\|\hat{A}_0\|_{(1)} + (n-1)\|\hat{D}\|_{(1)} \\
&= \|(n-2)\hat{A}_0 + (n-1)\hat{D}\|_{(1)} \\
&= \|\sum_{k=1}^{n}\hat{A}_k\|_{(1)} \leq \sum_{k=1}^{n}\|\hat{A}_k\|_{(1)} = \sum_{k=1}^{n}\|A_k\|_{(p)}^p \\
&\leq n\max_{1\leq k\leq n}\|A_k\|_{(p)}^p\ .
\end{aligned}$$

Taking p-th roots supplies

$$\max_{1\le k\le n} \|A_k\|_{(p)} \ge \left(\frac{n-2}{n}\right)^{1/p} \|A\|_{(p)}\ . \tag{17}$$

The matrix $A := E - I$ satisfies

$$\|A\|_{(p)} = \big(n(n-1)\big)^{1/p} \quad \text{and} \quad \|A_k\|_{(p)} = \big((n-1)(n-2)\big)^{1/p} \quad \text{for all } k$$

and thereby

$$\max_{1\le k\le n} \|A_k\|_{(p)} = \left(\frac{n-2}{n}\right)^{1/p} \|A\|_{(p)}\ .$$

Hence (17) is sharp. Taking limits gives

$$\lim_{p\to\infty} \left(\frac{n-2}{n}\right)^{1/p} = \begin{cases} 1 & \text{if } n > 2, \\ 0 & \text{if } n = 2, \end{cases}$$

and we conclude

$$\beta_n = \left(\frac{n-2}{n}\right)^{1/p} \quad \text{for } 1 \le p \le \infty \ \ (\text{with } 0^0 := 0 \text{ for } n = 2). \tag{18}$$

Summarizing, (16) and (18) prove Nr. 3 of Table 1. Note that $\|A\|_{(2)}$ is the Frobenius norm, so that (16) and (18) improve upon (11).

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