Entrywise lower and upper bounds for the Perron vector

S.M. Rump[a]

[a] *Technical University of Hamburg, Institute for Reliable Computing, Am Schwarzenberg-Campus 3, 21073 Hamburg, Germany*

and

Visiting Professor at Waseda University, Faculty of Science and Engineering, 3–4–1 Okubo, Shinjuku-ku, Tokyo 169–8555, Japan

Email: `rump@tuhh.de`

Abstract

Let an irreducible nonnegative matrix A and a positive vector x be given. Assume $\alpha x \leq Ax \leq \beta x$ for some $0 < \alpha \leq \beta \in \mathbb{R}$. Then, by Perron-Frobenius theory, α and β are lower and upper bounds for the Perron root of A. As for the Perron vector x^*, only bounds for the ratio $\gamma := \max_{i,j} x_i^*/x_j^*$ are known, but no error bounds against some given vector x. In this note we close this gap. For a given positive vector x and provided that α and β as above are not too far apart, we prove entrywise lower and upper bounds of the relative error of x to the Perron vector of A.

Key words: Perron-Frobenius theory, Perron vector, M-matrix
2010 MSC: 15A48, 15A42

1. Main result

Let $A = [A_{ij}] \in \mathbb{R}^{n \times n}$ be an irreducible nonnegative matrix. Then Perron-Frobenius Theory [5, Theorem 8.4.4] implies that the spectral radius $\varrho(A)$ of A is an algebraically simple eigenvalue, the Perron root, and there is a corresponding positive eigenvector. Often [5, Chapter 8.4] the positive vector x^* with the normalization $\|x^*\|_1 = 1$ is called "the" Perron vector; here we call a positive multiple of x^* "a" Perron vector.

Collatz' result [1] implies bounds for the Perron root, namely, for any positive vector $x \in \mathbb{R}^n$

$$\min_{1 \le i \le n} \frac{(Ax)_i}{x_i} \le \varrho(A) \le \max_{1 \le i \le n} \frac{(Ax)_i}{x_i} \tag{1}$$

with equalities if x is a Perron vector.

For x^* denoting a Perron vector with some normalization, several bounds [11, 12, 8, 7] are known for the ratio $\gamma := \max_{i,j} x_i^*/x_j^*$. For example,

$$\gamma \le \frac{\max_{i,j} A_{ij}}{\min_{i,j} A_{ij}} \tag{2}$$

is shown in [12] for positive $A = [A_{ij}]$. In [3] the bound

$$\gamma \le \left(\frac{\|A\|_\infty + \nu(A)}{\nu(A)} \right)^{n-1} . \tag{3}$$

for nonnegative irreducible A is given using the measure of irreducibility

$$\nu(A) := \min_{M \subseteq \{1,\ldots,n\}} \max_{i \in M, j \notin M} A_{ij} .$$

If some positive x with narrow left and right bounds in (1) is given, it is desirable to use this information for eigenvector bounds. Moreover, individual bounds for the entries of a Perron vector are preferable. In this note, inspired by [9], we close this gap. For a given positive vector x we develop entrywise lower and upper bounds for the relative error of x to a Perron vector of A.

We denote the set of real or complex $m \times n$ matrices by $M_{m,n}$, and use M_n if $m = n$. For $C \in M_n$ and $\mu \subseteq \{1, \ldots, n\}$ denote by $C[\mu] \in M_{|\mu|}$ the matrix consisting of the rows and columns of C in μ, and by $C[:, \mu] \in M_{n,|\mu|}$ the matrix with columns in μ. The identity matrix of dimension k is denoted by I_k, where the index is omitted if clear from the context. The matrix $|C|$ is the matrix of absolute values, and comparison of vectors or matrices is always entrywise.

We use the following. Let a Z-matrix C and positive vectors v, s be given, and suppose that a vector u satisfies $Cv \ge u > 0$. Then C is an M-matrix [2, Theorem 5.1], [6, Theorem 2.5.3.12], and (see [10, Theorem 3.7.7])

$$C^{-1}s \le C^{-1} \cdot \max_i \frac{s_i}{u_i} \cdot u \le \max_i \frac{s_i}{u_i} \cdot v. \tag{4}$$

Theorem 1. *Let A be an irreducible nonnegative $n \times n$ matrix. Suppose that $x = [x_i] \in \mathbb{R}^n$, $x > 0$ and $\|x\|_\infty = 1$. Suppose that $\alpha x \le Ax \le \beta x$, in which $0 < \alpha \le \beta$, and set $\delta := \beta - \alpha$. Let $k \in \{1, \dots, n\}$ be fixed but arbitrary, let $\mu := \{1, \dots, n\} \backslash \{k\}$, and let $x^* = [x_i^*]$ be the unique positive eigenvector of A such that $x_k^* = x_k$. Assume $\delta < x_k A_{ik}$ for all $i \in \mu$, then*

$$|x^* - x| \le \varepsilon \cdot x \quad \textit{with } \varepsilon := \max_{i \in \mu} \frac{\delta x_i}{x_k A_{ik} - \delta x_i} \,. \tag{5}$$

As a consequence, if k satisfies $x_k = \|x\|_\infty$ and $\delta < \min_{i \in \mu} A_{ik}$, then

$$|x^* - x| \le \varepsilon \cdot x \quad \textit{with } \varepsilon := \max_{i \in \mu} \frac{\delta}{A_{ik} - \delta} \,. \tag{6}$$

Proof. Denote $P := I[:, \mu]$ and by $e_{(k)} := I[:, k]$ the k-th column of the identity matrix. It follows that $PP^T + e_{(k)} e_{(k)}^T = I_n$, $P^T P = I_{n-1}$ and $P^T e_{(k)} = 0$. Moreover, $A[\mu] = P^T A P \in M_{n-1}$ and $x[\mu] = P^T x \in \mathbb{R}^{n-1}$.

Using $r := \varrho(A)$ for the spectral radius of A, Collatz's famous result (1) implies $\alpha \le r \le \beta$. We define $B := rI_{n-1} - A[\mu] \in M_{n-1}$ and $b := x_k P^T A e_{(k)} \in \mathbb{R}^{n-1}$. The entrywise monotonicity [14, Theorem 2.1] of the spectral radius of a nonnegative irreducible matrix implies $\varrho(A[\mu]) < r$, so that B is nonsingular.

It is known (see, e.g., [4, p. 3]) that $By = b$ implies that $z = Py + x_k e_{(k)}$ is a Perron vector of A with the normalization $z_k = x_k$. To confirm that write $ry = P^T A(Py + x_k e_{(k)})$ or $rP^T z = P^T A z$. Since $rI - A$ is singular, there is a nontrivial vector $q \in \mathbb{R}^n$ with $q^T(rI - A) = 0$. If $q_k = 0$, then $q = PP^T q$ and

$$0 = q^T PP^T (rI - A) P = q^T P (rI_{n-1} - P^T A P) = q[\mu]^T B$$

implies that B is singular, a contradiction. Hence $P^T(rI - A)z = 0$ gives

$$q^T (PP^T + e_{(k)} e_{(k)}^T)(rI - A)z = 0 = q_k e_{(k)}^T (rI - A)z$$

and $e_{(k)}^T (rI - A)z = 0$, and again using $P^T(rI - A)z = 0$ yields $(rI - A)z = 0$. Next

$$\begin{aligned} b - Bx[\mu] &= x_k P^T A e_{(k)} - (rP^T P - P^T A P) P^T x \\ &= P^T \left(x_k A e_{(k)} - rPP^T x + A(I - e_{(k)} e_{(k)}^T) x \right) \\ &= P^T \left(Ax - r(I - e_{(k)} e_{(k)}^T) x \right) \\ &= P^T (Ax - rx) \end{aligned} \tag{7}$$

and therefore

$$|b - Bx[\mu]| \le P^T \delta x. \tag{8}$$

Define $\underline{B} := \alpha I_{n-1} - A[\mu]$. Then $\underline{B}$ is a Z-matrix, and

$$\begin{aligned} \underline{B}x[\mu] &= (\alpha I_{n-1} - P^T AP)P^T x = P^T \left(\alpha x - A(I - e_{(k)} e_{(k)}^T) x\right) \\ &\ge P^T \left(x_k A e_{(k)} - \delta x\right) =: u. \end{aligned} \tag{9}$$

Now $\|x\|_\infty = 1$ implies that $u_i = \left(x_k A e_{(k)} - \delta x\right)_i \ge x_k A_{ik} - \delta > 0$ for all $i \in \mu$, so that $\underline{B}$ is an M-matrix [6, Theorem 2.5.3.12]. Therefore $\underline{B} \le B$ implies $0 \le B^{-1} \le \underline{B}^{-1}$ [6, Theorem 2.5.4], and (8) gives

$$|y - x[\mu]| = |B^{-1}(b - Bx[\mu])| \le \underline{B}^{-1} P^T \delta x.$$

Finally, using $y = P^T x^*$ and applying (4) with $C := \underline{B}$, $s = \delta x$, $v := x[\mu]$ and u as in (9) yields

$$|P^T(x^* - x)| = |y - x[\mu]| \le \max_{i \in \mu} \frac{\delta x_i}{x_k A_{ik} - \delta x_i} \cdot x[\mu]$$

and proves (5), from which (6) is obvious. □

Theorem 1 is applicable if all off-diagonal entries of at least one column (or row) is strictly less than the gap $\delta = \beta - \alpha$ of the bounds for the Perron root. Hence Theorem 1 is always applicable for positive A and small enough δ, for a cyclic shift permutation matrix or tridiagonal matrix with $n \ge 4$ it is not applicable. Some power iterations may be used to improve δ. More precisely [14, p. 34], $x^{(r)} := A^r x^{(0)}$ implies

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \ldots \le \varrho(A) \le \ldots \le \beta_2 \le \beta_1 \le \beta_0$$

for any positive $x^{(0)}$, where $\alpha_r := \min_i \left(\frac{x_i^{(r+1)}}{x_i^{(r)}} \right)$ and $\beta_r := \max_i \left(\frac{x_i^{(r+1)}}{x_i^{(r)}} \right)$. We close this note with two examples. Consider

$$A = \frac{1}{32} \begin{pmatrix} 24 & 4 & 68 \\ 4 & 21 & 29 \\ 0 & 3 & 11 \end{pmatrix}$$

with eigenvalues $(1, \frac{1}{2}, \frac{1}{4})$ and Perron vector $x^* = (1, \frac{7}{12}, \frac{1}{12})^T$. The estimate (6) is not applicable because $x_1^* = \|x^*\|_\infty$ but $A_{31} = 0$. However, (5) is applicable for sufficiently small δ. Starting with $x^{(0)} := (1,1,1)^T$ we calculate $x^{(r)}, \alpha_r, \beta_r$ as above for $r \in \{3,5,7,9\}$. Then, for $k = 2$ and for $k = 3$, we compute the entrywise relative error $\varepsilon_{k,r} := \max_{i\in\mu} \dfrac{\delta x_i}{x_k A_{ik} - \delta x_i}$ according to (5) for $x := x^{(r)}$ and $\delta := \beta_r - \alpha_r$. The results are displayed in Table 1.

Table 1: Entrywise relative error according to (5) of a Perron vector against $x := x^{(r)}$.

r	δ	$\varepsilon_{2,r}$	$\varepsilon_{3,r}$
3	0.023	0.48	0.21
5	0.0064	0.098	0.052
7	0.0021	0.030	0.017
9	0.00057	0.0079	0.0044

As can be seen, with increasing r the gap δ between the lower and upper bound of the Perron root becomes smaller, and the entrywise relative bounds for a Perron vector become better, as expected. The bounds $\varepsilon_{3,r}$ are superior because of the small entry A_{32}. The bound (2) on the maximum ratio $\gamma = \max_{i,j} x_i^*/x_j^*$ is not applicable because A is not positive, and (3) yields $\nu(A) = \frac{3}{32}$ and $\gamma \leq 1089$.

Finally we generate A by the Matlab command `A = rand(1000)`, so that the entries of the 1000×1000 matrix are uniformly distributed in $[0,1]$. We show in Table 2 the results of (5) and (6), both for the k with $x_k = \|x\|_\infty$.

Table 2: Entrywise relative error according to (5) of a Perron vector against $x := x^{(r)}$.

r	δ	ε by (5)	ε by (6)
3	$4.3 \cdot 10^{-4}$	-	-
5	$1.2 \cdot 10^{-7}$	$3.1 \cdot 10^{-4}$	$3.3 \cdot 10^{-4}$
7	$3.7 \cdot 10^{-11}$	$9.1 \cdot 10^{-7}$	$9.6 \cdot 10^{-7}$
9	$9.6 \cdot 10^{-13}$	$2.4 \cdot 10^{-9}$	$2.5 \cdot 10^{-9}$

The "-" for $r = 3$ means that, although δ is small, the condition $\delta < \min_{i\in\mu} A_{ik}$ is not satisfied for the k with $x_k = \|x\|_\infty$. Otherwise δ decreases more rapidly than for the first example, and practical experience suggests that this is not untypical. Note that the bound for ε computed by (6) is slightly weaker than that by (5).

After 9 power iterations, i.e., some $18n^2$ floating-point operations, the bounds for all entries of a Perron vector are accurate to about 9 decimal figures. Using the Matlab/Octave toolbox INTLAB [13] for reliable computing it is straightforward to compute mathematically correct lower and upper bounds for the Perron root and for a Perron vector.

Acknowledgment

The author wishes to thank the anonymous referees for their thorough reading and valuable comments which helped to improve readability of this note.

References

[1] L. Collatz. Einschließungssatz für die charakteristischen Zahlen von Matrizen. *Math. Z.*, 48:221–226, 1942.

[2] M. Fiedler. *Special matrices and their applications in numerical mathematics.* Martinus Nijhoff Publishers, Dordrecht, 1986.

[3] D. J. Hartfiel. Bounds for eigenvalues and eigenvectors of a nonnegative matrix which involve a measure of irreducibility. *SIAM J. Appl. Math.*, 24(1):6–8, 1973.

[4] D. Hartman, M. Hladík, and D. Říha. Computing the spectral decomposition of interval matrices and a study on interval matrix power, 2019. `arXiv:1912.05275v1`.

[5] R.A. Horn and Ch. Johnson. *Matrix Analysis.* Cambridge University Press, second edition, 2012.

[6] R.A. Horn, C.R. Johnson, Topics in matrix analysis, Cambridge University Press, 1991.

[7] Geoff A. Latham. A remark on Minc's maximal eigenvector bound for positive matrices. *SIAM J. Matrix Anal. Appl.*, 16(1):307–311, 1995.

[8] H. Minc. On the maximal eigenvector of a positive matrix. *SIAM J. Numer. Anal.*, 7(3):424–427, 1970.

[9] S. Miyajima. Fast verification for the Perron pair of an irreducible non-negative matrix. *Electronic Journal of Linear Algebra (ETNA)*, 37:402–415, 2021.

[10] A. Neumaier. *Interval Methods for Systems of Equations.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1990.

[11] A.M. Ostrowski. Bounds for the greatest characteristic root of a positive matrix. *J. London Math. Soc.*, 27:253–256, 1952.

[12] A.M. Ostrowski. On the eigenvector belonging to the maximal root of a non-negative matrix. *Proc. Edinburgh Math. Soc.*, 12:107–112, 1960.

[13] S.M. Rump. INTLAB - INTerval LABoratory. In Tibor Csendes, editor, *Developments in Reliable Computing*, pages 77–104. Kluwer Academic Publishers, Dordrecht, 1999. `http://www.ti3.tuhh.de/intlab`.

[14] R.S. Varga. *Matrix Iterative Analysis.* Prentice-Hall, Englewood Cliffs, N.J., 1962.