STRUCTURED PERTURBATIONS AND SYMMETRIC MATRICES

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Abstract. For a given n by n matrix the ratio between the componentwise distance to the nearest singular matrix and the inverse of the optimal Bauer-Skeel condition number cannot be larger than $(3 + 2\sqrt{2}) \cdot n$. In this note a symmetric matrix is presented where the described ratio is equal to n for the choice of most interest in numerical computation, for relative perturbations of the individual matrix components. It is shown that a symmetric linear system can be arbitrarily ill-conditioned, while any symmetric and entrywise relative perturbation of the matrix of less than 100% does not produce a singular matrix. That means that the inverse of the condition number and the distance to the nearest ill-posed problem can be arbitrarily far apart. Finally we prove that restricting structured perturbations to symmetric (entrywise) perturbations cannot change the condition number by more than a factor $(3 + 2\sqrt{2}) \cdot n$.

Key words. Structured perturbations, symmetric matrices, condition number.

0. Introduction. For a given nonsingular n by n matrix A, for $b \in \mathbf{R}^n$, $Ax = b$ and $(A + \delta A)\tilde{x} = b$, standard structured perturbation analysis [16] uses

$$(1) \qquad (I + A^{-1} \cdot \delta A) \cdot (x - \tilde{x}) = A^{-1} \cdot \delta A \cdot x.$$

For $|\delta A| \le E$, this motivates the definition of the Bauer-Skeel condition number ([2], [3], [14], [15])

$$\operatorname{cond}_{BS}(A, E) := \| \, |A^{-1}| \cdot E \, \| \, .$$

Here and in the following, absolute value and comparison of matrices is always to be understood *entrywise*. It is well known ([2], Lemma 1, see also [5]) that for any norm being subordinate to an absolute vector norm (ρ denotes the spectral radius)

$$\inf_D ||D^{-1}BD|| = \rho(B) \quad \text{for} \quad B \ge 0,$$

where the infimum is taken over all nonsingular diagonal matrices. Therefore, we define the minimum componentwise condition number subject to a nonnegative weight matrix E by

$$\operatorname{cond}(A, E) = \rho(|A^{-1}| \cdot E).$$

This optimal Bauer-Skeel condition number has been used by Demmel [4]. It is independent of row and column scaling, and Bauer observed in [2] that for relative perturbations ($E = |A|$) this is equal to the minimum achievable (traditional) condition number over row and column scaling:

$$\min_{D_1, D_2} \operatorname{cond}_\infty(D_1 A D_2) = \rho(|A^{-1}| \cdot |A|),$$

with $\operatorname{cond}_\infty(A) = ||A||_\infty \cdot ||A^{-1}||_\infty$. So for suitably scaled matrix and relative perturbations, the normwise and componentwise condition number are the same. Define the componentwise distance to the nearest singular matrix by

$$(2) \qquad \sigma(A, E) := \min\{\alpha \ge 0 \mid \exists \, \tilde{E} \text{ with } |\tilde{E}| \le \alpha \cdot E \text{ and } A + \tilde{E} \text{ singular}\}.$$

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If no such α exists set $\sigma(A, E) := \infty$. Then there is a well known relation (see, for example, [10]) between the componentwise condition number and $\sigma(A, E)$. For nonsingular A and singular $A + \tilde{E} = A(I + A^{-1}\tilde{E})$ with $|\tilde{E}| \leq \alpha \cdot E$ it is

$$1 \leq \rho(A^{-1}\tilde{E}) \leq \rho(|A^{-1}\tilde{E}|) \leq \alpha \cdot \rho(|A^{-1}| \cdot E),$$

and therefore

$$\frac{1}{\text{cond}(A, E)} = \frac{1}{\rho(|A^{-1}|E)} \leq \sigma(A, E).$$

For the normwise distance to the nearest singular matrix the well known theorem by Gastinel and Kahan says that (see [8], Theorem 6.5)

$$\min\{ \; ||\Delta|| \; | \; A + \Delta \text{ singular} \; \} = ||A^{-1}||^{-1} = \frac{||A||}{\text{cond}(A)}.$$

In other words, for any ill-conditioned matrix there exists a singular matrix not too far away in a normwise sense. The natural question is whether this is also true in a componentwise sense, i.e., does there exist a singular matrix not too far away in a *componentwise* sense? In other words, are there finite constants $\gamma(n)$ with $\sigma(A, E) \leq \gamma(n)/\rho(|A^{-1}| \cdot E)$?

N. J. Higham and Demmel conjectured in 1992 that such constants $\gamma(n)$ exist for relative perturbations $E = |A|$. Higham writes [8]: *"This conjecture is both plausible and aesthetically pleasing because $\sigma(A, E)$ is invariant under two-sided diagonal scalings of A and $\rho(|A^{-1}| \cdot |A|)$ is the minimum ∞-norm condition number achievable by such scalings"*. The conjecture has been solved in the affirmative in [13], see also [?], for arbitrary nonnegative weight matrices:

$$(3) \qquad\qquad \frac{1}{\rho(|A^{-1}| \cdot E)} \leq \sigma(A, E) \leq \frac{(3 + 2\sqrt{2}) \cdot n}{\rho(|A^{-1}| \cdot E)}.$$

Furthermore, better constants than $3 + 2\sqrt{2}$ have been given for certain n, and it has been shown that the upper bound in (3) cannot hold with a constant less than 1. The inequalities (3) imply:

> For a suitably scaled matrix A with nonnegative weight matrix E, the componentwise distance to the nearest singular matrix $\sigma(A, E)$ and the inverse componentwise condition number $\text{cond}(A, E)^{-1}$ cannot differ by more than a factor $(3 + 2\sqrt{2}) \cdot n$.
>
> A matrix that is ill-conditioned in the componentwise sense (that is, having a large $\text{cond}(A, E)$) is componentwise near to a singular matrix.

In practical problems frequently structured data occur, that is (componentwise) perturbations of the data are not independent of each other. A simple dependency is symmetry and leads to the important class of symmetric matrices.

In this note we investigate various aspects of symmetric structured pertubations. There are few papers in the literature restricting perturbations to symmetric perturbations. In [9], Jansson considers symmetric linear systems, the data of which are afflicted with tolerances. He gives an algorithm for computing bounds for the solution of all linear systems with data within the tolerances, where the componentwise perturbations are structured to allow only symmetric (or skew-symmetric) matrices.

Jansson shows that the solution set for those structured perturbations may be much smaller than the solution set with general componentwise perturbations.

Independent of this work, Higham and Higham [7] derive a generalized error analysis for linear systems with arbitrary linear dependencies among the matrix elements. An algorithm for calculating an inclusion of the solution set subject to arbitrary linear dependencies between the matrix elements is given in [12]. Linear dependencies cover symmetric, persymmetric, Toeplitz and other classes of matrices.

In [6], D. J. Higham derives relations between general and symmetric condition numbers subject to normwise perturbations. Higham shows that both measures are equal or at least not far apart for symmetric as well as for unsymmetric matrices. In this paper we show that this is also true for componentwise perturbations.

The paper is organized as follows. In the first section we derive a general $n \times n$ example of a symmetric matrix A with relative perturbations ($E = |A|$) showing that the factor $3 + 2\sqrt{2}$ in the second inequality of (3) cannot be replaced by a factor less than 1.

In the second section a structured symmetric condition number is derived and it is shown that a large (symmetric) condition number does not imply that a singular matrix is nearby subject to componentwise symmetric perturbations. That means for the problem of matrix inversion, the inverse of the condition number and the distance to ill-posedness can be arbitrarily far apart when perturbations are restricted to symmetric structured perturbations.

In the third section we show relations between the general and structured symmetric condition number of a matrix. It is shown that for any weight matrix the ratio between those two condition numbers ist between 1 and $(3 + 2\sqrt{2}) \cdot n$.

1. A symmetric matrix with large $\gamma(n)$. Following we give a general $n \times n$ example of a *symmetric* matrix such that for relative perturbations

$$\sigma(A, |A|) = \frac{n}{\rho(|A^{-1}| \cdot |A|)}.$$

Consider the symmetric tridiagonal matrix

$$(4) \qquad A = \begin{pmatrix} 1 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & & \ddots & \\ & & & 1 & s \end{pmatrix}$$

with $s = (-1)^{n+1}$. There are all $1's$ in the super- and subdiagonal, 1 in the $(1,1)$-component and s in the (n,n)-component. The connectivity graph of the matrix (see [17]) is

Thus for each nonzero element A_{ij} of A, there is *exactly one* permutation $\pi : \{1, \cdots, n\} \to \{1, \cdots, n\}$ with $\prod_{i=1}^{n} A_{i,\pi(i)} \neq 0$ and $\pi(i) = j$. Those products are clearly of absolute value 1. Therefore, the expansion of the determinant of any $(n-1) \times (n-1)$

submatrix of A shows that all minors of A are equal to 1 in absolute value, and therefore

$$(5) \qquad |A^{-1}| = (\det A)^{-1} \cdot (1)_{nn},$$

the matrix of all $1's$ divided by the determinant of A. With the same argument one sees that there are *exactly two* permutations π, τ with nonzero value of the products

$$c := \prod_{i=1}^{n} A_{i,\pi(i)} \quad \text{and} \quad d := \prod_{i=1}^{n} A_{i,\tau(i)},$$

namely (in cycle notation)

$$\pi = (12)(34)(56)\cdots \quad \text{and} \quad \tau = (1)(23)(45)\cdots$$

Carefully looking at the signs reveals that s is in (4) correctly chosen such that for every n, $|\det A| = |c + d| = 2$. Hence, any relative change less than 100% of the nonzero elements of A cannot produce a singular matrix, and therefore

$$\sigma(A, |A|) = 1.$$

In view of (5), using $|\det A| = 2$ and the fact that each column of $|A|$ consists of exactly two nonzero entries, both equal to 1, we have

$$|A^{-1}| \cdot |A| = (1)_{nn}.$$

This means $\rho(|A^{-1}| \cdot |A|) = n$, and combining this with $\sigma(A, |A|) = 1$ yields the following result.

THEOREM 1.1. *For the n by n symmetric matrix A defined by (4), the underestimation by the inverse optimal condition number $\rho(|A^{-1}| \cdot |A|)^{-1}$ of the componentwise distance to the nearest singular matrix $\sigma(A, |A|)$ subject to relative perturbations of the matrix components $(E = |A|)$ is equal to n:*

$$\sigma(A, |A|) = \frac{n}{\rho(|A^{-1}| \cdot |A|)}.$$

In [?], [13] we conjectured that this is the maximum possible underestimation.

A general drawback of condition numbers is that they do not reflect dependencies on the matrix elements. Componentwise perturbation analysis is more versatile than traditional condition numbers, but nevertheless the perturbations are assumed to be entrywise independent.

2. The symmetric condition number and distance to ill-posedness. Consider a linear system $Ax = b$ with nonnegative weight matrix E and nonnegative weight vector e. Assume all matrices $\tilde{A}$ with $|\tilde{A} - A| \leq E$ to be nonsingular. Then the solution set of the perturbed systems is

$$(6) \qquad \Sigma_{E,e}(A, b) := \{\tilde{A}^{-1} \cdot \tilde{b} \mid |\tilde{A} - A| \leq E \text{ and } |\tilde{b} - b| \leq e\}.$$

A vector x belongs to the solution set if and only if it is a feasible point (without sign restriction) to the set of inequalities/equalities

$$\begin{array}{ll} \tilde{A} - E \leq A & -\tilde{A} - E \leq -A \\ \tilde{b} - e \leq b & -\tilde{b} - e \leq -b \\ \tilde{A}x = \tilde{b}. & \end{array}$$

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Using suitable LP-problems it follows (see [10])

(7)

i)   the solution set $\Sigma_{E,e}(A, b)$ is convex in every orthant,

ii)   the extreme points of $\Sigma_{E,e}(A, b)$ are of the form
$$(A + \tilde{E})^{-1} \cdot (b + \tilde{e}) \text{ with } |\tilde{E}| = E, |\tilde{b}| = b,$$

iii) $\text{ch}(\Sigma_{E,e}(A, b)) = \text{ch}\{(A + \tilde{E})^{-1} \cdot (b + \tilde{e}) \mid |\tilde{E}| = E, |\tilde{b}| = e\},$

where ch denotes the convex hull. Those statements are true for componentwise and mutually *independent* perturbations of A, b subject to the weights E, e.

The most simple and important dependency among the matrix elements is symmetry. Assume the weight matrix to be symmetric, $E = E^T$, but not necessarily the matrix A. For componentwise but structured perturbations the statements (7) are no longer true. Similar to (6) define the symmetric solution set

$$\Sigma_{E,e}^{sym}(A, b) := \{\tilde{A}^{-1} \cdot \tilde{b} \mid |\tilde{A} - A| \le E, |\tilde{b} - b| \le e \text{ and } (\tilde{A} - A)^T = \tilde{A} - A\}.$$

If A is unsymmetric, this is the set of solutions of symmetrically perturbed linear systems. The symmetric solution set need not be convex within an orthant, and the convex union of solutions $\tilde{A}^{-1} \cdot \tilde{b}$ with $|\tilde{A} - A| = E$, $(\tilde{A} - A)^T = \tilde{A} - A$ and $|\tilde{b} - b| = e$ need not be a superset of $\Sigma_{E,e}^{sym}(A, b)$. As an example consider

$$A = \begin{pmatrix} -2 & 1 \\ 1 & 1.1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

with perturbations $E = 0.8 \cdot (1)_{22}$ of the matrix, no perturbations ($e = 0$) of the right hand side. Then the symmetric solution set looks as follows (for further discussions concerning the symmetric solution set see also [9, 1]).
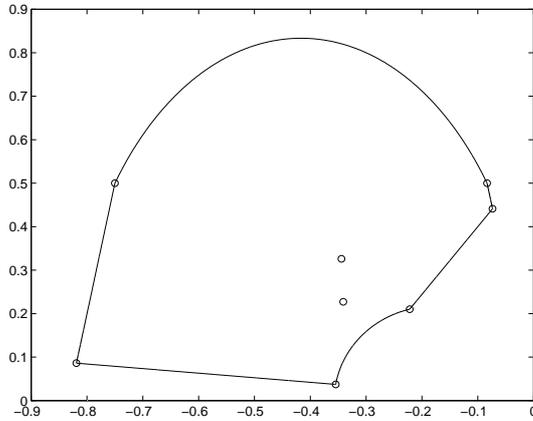

FIG. 2.1. *Symmetric solution set* $\Sigma_{E,e}^{sym}(A, b)$.

The circles are the solutions of the 8 linear systems $(A + \tilde{E})^{-1} \cdot b$ with $|\tilde{E}| = E$, $\tilde{E} = \tilde{E}^T$.

In view of the identity (1) we define the *symmetric structured condition number* by

(8) $$\text{cond}_{sym}(A, E) := \max_{\substack{|\tilde{E}| \le E \\ \tilde{E}^T = \tilde{E}}} \rho(|A^{-1} \cdot \tilde{E}|) = \max_{\substack{|\tilde{E}| \le E \\ \tilde{E}^T = \tilde{E}}} \inf_D ||D^{-1} A^{-1} \tilde{E} D||_\infty.$$

The second equality, follows by $\rho(|B|) = \inf_D ||D^{-1}BD||_\infty$. For unsymmetric E, define E^{sym} by $E_{ij}^{sym} := \min(E_{ij}, E_{ji})$. Then $\text{cond}_{sym}(A, E) = \text{cond}_{sym}(A, E^{sym})$, and it is no loss of generality to assume E to be symmetric. Note that A is not required to be symmetric.

According to the previous discussion we note that the inequalities $|\tilde{E}| \leq E$ in (8) cannot be replaced by equalities. An example is

$$A = \begin{pmatrix} 17 & 53 & 6 \\ 53 & -18 & 24 \\ 6 & 24 & 11 \end{pmatrix} \quad \text{with} \quad E = |A|,$$

with

$$\max_{\substack{|\tilde{E}| = E \\ \tilde{E} = \tilde{E}^T}} \rho(|A^{-1}\tilde{E}|) = 2.914,$$

but

$$\rho(|A^{-1}\tilde{\tilde{E}}|) > 2.923 \quad \text{for} \quad \tilde{\tilde{E}} = \begin{pmatrix} -17 & 53 & -0.176 \\ 53 & -18 & -24 \\ -0.176 & -24 & 11 \end{pmatrix}.$$

Like in the classical result by Bauer and Skeel (Theorem 2.14 in [16]), (1) implies for componentwise symmetric perturbations and suitably scaled A, E with $\kappa := \text{cond}_{sym}(A, E)$,

$$||x - \tilde{x}||_\infty \leq \epsilon \cdot \frac{\kappa}{1 - \epsilon \cdot \kappa} \cdot ||x||_\infty \quad \text{for} \quad |\delta A| \leq \epsilon \cdot E, \quad \delta A = \delta A^T.$$

Like the symmetric structured condition number, we define the *symmetric structured distance to the nearest singular matrix* by

$$\sigma_{sym}(A, E) := \min\{\alpha \geq 0 \mid \exists \tilde{E} \text{ with } |\tilde{E}| \leq \alpha \cdot E, \tilde{E} = \tilde{E}^T \text{ and } A + \tilde{E} \text{ singular}\}.$$

We set $\sigma_{sym}(A, E) := \infty$ if no such α exists. Like for the symmetric structured condition number, we may assume without loss of generality $E = E^T$. Note that again A is not assumed to be symmetric.

The symmetric and (general) componentwise distance to the nearest singular matrix may be arbitrarily far apart. For

$$A = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

it is $\sigma(A, E) = \epsilon$, but no symmetric perturbation $\tilde{E}$, $|\tilde{E}| \leq \alpha \cdot E$, whatsoever can produce a singular matrix, hence $\sigma_{sym}(A, E) = \infty$. The symmetric condition number is $\text{cond}_{sym}(A, E) = \epsilon^{-1}$.

The question arises whether this situation changes when assuming A to be symmetric with relative perturbations, $A = A^T$, $E = |A|$. Is it true that for an ill-conditioned symmetric matrix (subject to symmetric relative perturbations, that is, having large $\text{cond}_{sym}(A, E)$), a small (symmetric) relative perturbation produces a singular matrix (that is, $\sigma_{sym}(A, E)$ is small)?

Unfortunately, the corresponding result (3) for *general* componentwise perturbations does not carry over to symmetric ones. Consider

$$(9) \qquad A = A_\epsilon := \begin{pmatrix} \epsilon & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix} \qquad \text{and} \quad E = |A|.$$

Then

$$\tilde{A} = \begin{pmatrix} (1+\delta_1)\epsilon & 1+\delta_2 & 0 & 1+\delta_7 \\ 1+\delta_2 & 0 & 1+\delta_3 & 0 \\ 0 & 1+\delta_4 & 0 & 1+\delta_5 \\ 1+\delta_7 & 0 & 1+\delta_5 & -1+\delta_6 \end{pmatrix},$$

is a relative perturbation of A not larger than $\max_i |\delta_i|$. For $\delta_3 = \delta_4$ it is a *general* symmetric relative perturbation. A calculation yields

$$(10) \qquad \begin{aligned} \det \tilde{A} &= (c - \delta_3 \cdot d)(c - \delta_4 \cdot d) + \epsilon \cdot (1+\delta_1)(1+\delta_3)(1+\delta_4)(1-\delta_6) \\ \text{with} \quad & c = \delta_2 + \delta_5 + \delta_2\delta_5 - \delta_7 \quad \text{and} \quad d = 1 + \delta_7. \end{aligned}$$

For a *general* symmetric relative perturbation this means with $\delta_3 = \delta_4 =: \delta$,

$$\det \tilde{A} = (c - \delta \cdot d)^2 + \epsilon \cdot (1+\delta_1)(1+\delta)^2(1-\delta_6).$$

Hence, for $\epsilon > 0$ the perturbed matrix $\tilde{A}$ is nonsingular for any symmetric relative perturbation less than 1:

(11) *For $\epsilon > 0$, any relative symmetric perturbation of A_ϵ less than 1 is nonsingular, that is $\sigma_{sym}(A_\epsilon, |A_\epsilon|) = 1$.*

Define a specific *un*symmetric perturbation by $\delta_1 = \delta_2 = \delta_5 = \delta_6 = \delta_7 = 0$ and $\delta_3 = -\delta_4 =: \delta$. Then by (10), $c = 0$ and $\det \tilde{A} = -\delta^2 + \epsilon \cdot (1 - \delta^2)$, or $\det \tilde{A} = 0$ for $\delta = (\frac{\epsilon}{1+\epsilon})^{1/2}$. Hence, for unsymmetric perturbations,

$$\sigma(A_\epsilon, |A_\epsilon|) < \epsilon^{1/2} \quad \text{for any} \quad \epsilon > 0.$$

Furthermore, a computation yields $\text{cond}_{sym}(A_\epsilon, |A_\epsilon|) > \epsilon^{-1/2}$. Finally, we give a linear system with matrix A_ϵ which is ill-conditioned subject to relative (symmetric) perturbations of the matrix. It is $\det(A_\epsilon) = \epsilon$, and for the right hand side $b = (3\epsilon, 2, -2\epsilon, 3 - \epsilon)^T$ it is

$$b = (A_\epsilon + \delta A) \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = A_\epsilon \cdot \begin{pmatrix} 5 \\ 1 - 3\epsilon \\ -3 \\ -1 + \epsilon \end{pmatrix} \quad \text{for} \quad \delta A = \epsilon \cdot \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix},$$

showing that the linear system is truly ill-conditioned. The same argument can be applied for $n > 4$ by augmenting the matrix A_ϵ by an identity matrix in the lower right corner.

THEOREM 2.1. *For $n \geq 4$ and any $\epsilon > 0$ there exist symmetric matrices A with the following properties:*

i) Any symmetric relative perturbation of A less than 1 does not produce a singular matrix: $\sigma_{sym}(A, |A|) = 1$.

ii) The matrix is ill-conditioned subject to symmetric perturbations, that is, there exists a right hand sides b such that a symmetric relative perturbation of A of size ϵ imposes a relative change of components of the solution of $Ax = b$ by more than 1: $\text{cond}_{sym}(A_\epsilon, |A_\epsilon|) > \epsilon^{-1/2}$.

Therefore, the inverse condition number and the distance to the nearest ill-posed problem can be arbitrarily far apart when perturbations are restricted to symmetric perturbations.

The reason for this behavior is that the space of admissible perturbations is restricted to symmetric ones, and relative symmetric changes move A basically towards well-conditioned matrices. For the case of general perturbations this is not true as demonstrated by (3).

3. The ratio between $\text{cond}(A, E)$ **and** $\text{cond}_{sym}(A, E)$**.** In the example (9), there is no singular matrix nearby in a componentwise sense, but the matrix is ill-conditioned with respect to general *and* with respect to symmetric perturbations. In fact, $\text{cond}(A_\epsilon, |A_\epsilon|) \approx \text{cond}_{sym}(A_\epsilon, |A_\epsilon|) \approx \sigma(A_\epsilon, |A_\epsilon|)^{-1} \approx 4 \cdot \epsilon^{-1/2}$. The question remains, whether it is possible that a matrix is ill-conditioned with respect to general componentwise perturbations, but well-conditioned with respect to componentwise symmetric perturbations. In other words, what is the relation between $\text{cond}(A, E)$ and $\text{cond}_{sym}(A, E)$ for symmetric weight matrix E?

The trivial inequality is $\text{cond}(A, E) \geq \text{cond}_{sym}(A, E)$. For the other inequality, define a signature matrix S to be diagonal with $S_{ii} \in \{+1, -1\}$, so that $|S| = I$. The *real* spectral radius is defined by

$$\rho_0(A) := \max\{|\lambda| \mid \lambda \text{ a } \textit{real} \text{ eigenvalue of } A\}.$$

If the spectrum of A does not contain a real eigenvalue, we define $\rho_0(A) := 0$. It has been shown by Rohn [11] that

$$(12) \qquad \max_{S_1, S_2} \rho_0(S_1 A^{-1} S_2 E) = \sigma(A, E)^{-1},$$

with the maximum taken over signature matrices S_1, S_2, and including the case $0 = 1/\infty$. For the signature matrices S_1, S_2 attaining the maximum in (12) it follows for $E = E^T$,

$$
\begin{aligned}
\text{cond}_{sym}(A, E) &= \max\{\rho(|A^{-1}\tilde{E}|) \mid |\tilde{E}| \leq E \text{ and } \tilde{E}^T = \tilde{E}\} \\
&\geq \rho(|A^{-1} S_2 E S_2|) \\
&= \rho(|S_1 A^{-1} S_2 E|) \\
&\geq \rho_0(S_1 A^{-1} S_2 E) \\
&= \sigma(A, E)^{-1},
\end{aligned}
$$

where the first inequality follows because with E also $S_2 E S_2$ is symmetric, the second equality follows because $|A^{-1} S_2 E S_2| = |A^{-1} S_2 E| = |S_1 A^{-1} S_2 E|$, and the following is a consequence of Perron-Frobenius theory. Note that only the weight matrix E is required to be symmetric, not the matrix A. Combining the results with (2) and (3) yields the following theorem.

THEOREM 3.1. *For any nonsingular (not necessarily symmetric) $n \times n$ matrix A and symmetric nonnegative $n \times n$ weight matrix E,*

$$\text{cond}(A, E) \geq \text{cond}_{sym}(A, E) \geq \frac{\text{cond}(A, E)}{(3 + 2\sqrt{2}) \cdot n}.$$

In words:

Restricting componentwise perturbations to componentwise symmetric perturbations cannot decrease the condition number by more than a factor $(3 + 2\sqrt{2}) \cdot n$.

Finally we mention that a linear system may be well-conditioned for symmetric perturbations and a (very) *specific* right hand side while it is arbitrarily ill-conditioned for general perturbations. Consider

$$A = \begin{pmatrix} 1 & 1 + \epsilon \\ 1 + \epsilon & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 2 + \epsilon \\ 2 + \epsilon \end{pmatrix} \quad \text{and} \quad A^{-1}b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then $A + \tilde{E}$ is nonsingular for any $|\tilde{E}| < \epsilon \cdot E$. For symmetric componentwise perturbations we have for any $|\tilde{E}| \leq \epsilon(1 - \epsilon) \cdot E$ with $\tilde{E} = \tilde{E}^T$,

$$(A + \tilde{E})^{-1}b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e \quad \text{with} \quad |e| \leq \begin{pmatrix} \epsilon/2 \\ \epsilon/2 \end{pmatrix} + \mathrm{O}(\epsilon^2),$$

while a specific *un*symmetric perturbation produces

$$(A + \tilde{E})^{-1} \cdot b = \begin{pmatrix} 2 \\ \epsilon \end{pmatrix} + \mathrm{O}(\epsilon^2) \quad \text{for} \quad \tilde{E} = \begin{pmatrix} 0 & \epsilon(1 - \epsilon) \\ -\epsilon(1 - \epsilon) & 0 \end{pmatrix}.$$

Note that for *general* right hand side the matrix is ill-conditioned for symmetric *and* unsymmetric structured perturbations: $\mathrm{cond}(A, E) = \mathrm{cond}_{sym}(A, E) = \epsilon^{-1}$. Summarizing:

For a suitably scaled matrix with arbitrary weight matrix E,
  the normwise condition number is equal to
  the componentwise condition number, and
  the symmetric componentwise condition number
cannot differ from both by more than a factor $(3 + 2\sqrt{2}) \cdot n$.

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