ADDENDUM TO THE DETERMINANT OF A PERTURBED IDENTITY MATRIX*

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Abstract. Many bounds for the determinant $\det(I+E)$ of a perturbed identity matrix are known. Mostly, the upper bound is inferior to the classical Hadamard bound. In this note we give simple and efficiently computable relative bounds differing by $\|E\|_F^3$, where the upper bound is usually better than Hadamard's bound.

Key words. Determinant, Hadamard bound, Hans-Schneider bound, perturbation of identity, $\mathcal{O}(\varepsilon^3)$

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1. Introduction and main result. Let E be a real or complex $n \times n$ matrix. The classical Hadamard bound reads

$$|\det(I+E)| \le \prod_{k=1}^{n} \|(I+E)_{k*}\|_2 =: H\,, \tag{1}$$

where I denotes the identy matrix of appropriate size. The computational effort to compute the bound is $\mathcal{O}(n^2)$ operations. Brent et al. [1, 2] gave sharp lower and upper bounds depending on the maximal absolute values of the diagonal and off-diagonal elements of E. The upper bounds are based on Hadamard's bound and therefore inferior to it.

Denoting by $\epsilon := \|E\|_F = [\mathrm{tr}(E^H E)]^{1/2}$ the Frobenius (or Hilbert-Schmidt) norm and assuming that the spectral radius $\rho(E)$ of E is strictly less than $1-\epsilon^2/2$, we recently [5] proved

$$\left|\frac{\det(I+E)-\exp(\mathrm{tr}(E))}{\exp(\mathrm{tr}(E))}\right| \le \frac{\epsilon^2}{2(1-\epsilon-\epsilon^2/2)}. \tag{2}$$

However, we also showed that the implied upper bound is always inferior to (1) except for E being the null matrix.

Suppose that the diagonal of E is zero, and denote $s_i := \sum_{j=1}^{n} |E_{ij}|$ for $1 \le i \le n$ and $S := \sum_{i=1}^{n} s_i$. Then Hans Schneider [6] proved

$$|\det(I+E)| \le \frac{e^{-S}}{\prod_{i=1}^{n}(1-s_i)} =: \mu$$

provided that $s_i < 1$ for all i. That upper bound can also not be better than the classical Hadamard bound. To see this use

$$\left(\frac{e^{-x}}{1-x}\right)^2 \ge \frac{(1-x+x^2/2-x^3/6)^2}{(1-x)^2} \ge 1+\frac{2(x^2/2-x^3/6)}{1-x} \ge 1+x^2 \qquad \text{for} \quad 0 \le x < 1,$$

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so that

$$\mu^2 = \prod_{i=1}^{n} \left(\frac{e^{-s_i}}{1-s_i} \right)^2 \geq \prod_{i=1}^{n} (1+s_i^2) = \prod_{i=1}^{n} (1 + (\sum_{j=1}^{n} |E_{ij}|)^2) \geq \prod_{i=1}^{n} \sum_{j=1}^{n} |(I+E)_{ij}|^2 = H^2.$$

To compute any of the mentioned bounds requires $\mathcal{O}(n^2)$ operations. Exploiting Fredholm's identity [3]

$$\det(I+E) = \exp\left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\operatorname{tr}(E^k)}{k} \right) \tag{3}$$

an upper bound of $|\det(I+E)|$ can be derived to any desired accuracy, however, the computational cost increases to $\mathcal{O}(n^3)$ operations. Following the proof of (2) we derive a bound of quality $\mathcal{O}(\epsilon^3)$ requiring $\mathcal{O}(n^2)$ operations which is usually better than Hadamard's bound (1).

THEOREM 1.1. *Let E be a real or complex $n \times n$ matrix and suppose $\rho(E) < (1+\epsilon^2/3)^{-1}$. Then*

$$\left| \frac{\det(I+E)}{\exp\left(\operatorname{tr}(E) - \operatorname{tr}(E^2)/2\right)} - 1 \right| \leq \frac{\epsilon^2 \rho(E)}{3(1-\rho(E)) - \epsilon^2 \rho(E)} \leq \frac{\epsilon^3}{3(1-\epsilon) - \epsilon^3}. \tag{4}$$

PROOF. Let λ_k denote the eigenvalues of E. Then $|\lambda_k| \leq \rho(E) < 1$, and abbreviating $D := \operatorname{tr}(E) - \dfrac{\operatorname{tr}(E^2)}{2}$ implies

$$\det(I+E) = \exp\left(\sum_{k=1}^{n} \log(1+\lambda_k) \right) = \exp\left(D + \sum_{k=1}^{n} \lambda_k^2 \left(\sum_{j=1}^{\infty} \frac{(-1)^{j+1} \lambda_k^j}{j+2} \right) \right) =: \exp(D + \Phi).$$

Furthermore,

$$|\Phi| \leq \sum_{k=1}^{n} \frac{|\lambda_k|^2}{3} \left(\sum_{j=1}^{\infty} |\lambda_k|^j \right) \leq \frac{\epsilon^2}{3} \cdot \frac{\rho(E)}{1-\rho(E)} =: \Psi < 1$$

by $\rho(E) < (1+\epsilon^2/3)^{-1}$. Hence, [4, 4.5.16] $|e^z - 1| \leq e^{|z|} - 1$ for $z \in \mathbb{C}$ and [4, 4.5.11] $e^x - 1 \leq \frac{x}{1-x}$ for $x < 1$ give

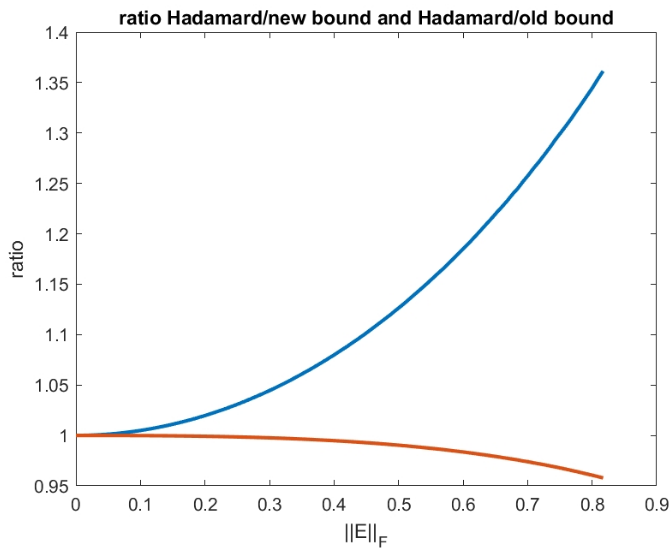
$$\left| \frac{\det(I+E)}{\exp(D)} - 1 \right| = |\exp(\Phi) - 1| \leq \exp(|\Phi|) - 1 \leq \frac{|\Phi|}{1-|\Phi|} \leq \frac{|\Psi|}{1-|\Psi|} = \frac{\epsilon^2 \rho(E)}{3(1-\rho(E)) - \epsilon^2 \rho(E)}.$$

□

The additional effort to obtain the bound (4) rather than (2) is to compute the trace of E^2. That amounts to $\mathcal{O}(n^2)$ operations, so basically doubles the effort. However, that is still negligible compared to the usual $\mathcal{O}(n^3)$ operations to compute the determinant.

Note that (4) yields relative lower and upper bounds of $\det(I+E)$ for real or complex E, not only for $|\det(I+E)|$. To assert the quality of the implied upper bound on $|\det(I+E)|$ we generated 1000 random test matrices with each entry drawn from a standard normal distribution and with given Frobenius norm. We then plot (the upper curve) the median of the ratio of Hadamard's bound (1) and the upper bound in (4). For comparison, we also plot (lower curve) the ratio of Hadamard's bound (1) and the previous upper bound in (2).

As can be seen in the figure, on the average, Hadamard's bound becomes up to 35% weaker than the bound in (4) for increasing Frobenius norm of E, whereas Hadamard's bound is up to 5% stronger than (2).



Note that this is based on random samples. Occasionally, Hadamard's bound on $|\det(I+E)|$ is better than (4), however, we could not identify conditions for that.

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