

On Generalized Schur Algorithms

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Classification and normal forms of functions

Schur functions

Jacobi transformation

Cayley transform

Carathéodory functions

Matrix decomposition

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- Matrix decomposition

Schur algorithm; modern form

- Reformulation of Schur's expansion

- Displacement structure

- Cholesky decomposition

- The Schur algorithm

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- Displacement structure
- Fundamental properties
- A generalized Schur algorithm

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It turns out that the coefficients s_k of the power series are **not** that useful when investigating Schur functions. Instead, the so-called **Schur coefficients** better describe their properties, and thus, play the dominant rôle.

Expansion of Schur functions

Schur used that for $\alpha \in \mathbb{C}$ with $|\alpha| < 1$ the “linear” transformation (**Blaschke factor**, no normalization; Moebius transformation)

$$\mathfrak{B}(z) = \frac{z - \alpha}{1 - \bar{\alpha}z} \quad (2)$$

maps \mathbb{D} to itself.

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$$s_{i+1}(z) = \frac{1}{z} \frac{s_i(z) - \gamma_i}{1 - \bar{\gamma}_i s_i(z)}, \quad \gamma_i = s_i(0). \quad (4)$$

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Then all $\{s_i\}_{i=0}^m$ satisfy $|s_i(z)| \leq 1$. The constants γ_i are the aforementioned **Schur coefficients** or **reflection coefficients**.

Rational expansion of Schur functions

Schur developed recursions for the expansion by writing the function s in terms of a rational function,

$$s(z) = \frac{a(z)}{b(z)} = \frac{\sum_{k=0}^{\infty} a_k z^k}{\sum_{k=0}^{\infty} b_k z^k}, \quad b_0 \neq 0. \quad (5)$$

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$$g_i(z) = \sum_{k=0}^{\infty} a_{i+k} z^k \quad \text{and} \quad h_i(z) = \sum_{k=0}^{\infty} b_{i+k} z^k \quad (7)$$

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and gave determinantal expressions for $D_i(z)$ and $\Delta_i(z)$ (the next two pages ...)

Explicit solution of the recursion

(remember that $g_i(z) = \sum_{k=0}^{\infty} a_{i+k}z^k$ and $h_i(z) = \sum_{k=0}^{\infty} b_{i+k}z^k$)

$$D_i(z) = \begin{vmatrix} 0 & 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_{i-1} & g_i(z) \\ \bar{b}_0 & 0 & \cdots & 0 & 0 & a_0 & \cdots & a_{i-2} & g_{i-1}(z) \\ \bar{b}_1 & \bar{b}_0 & \cdots & 0 & 0 & 0 & \cdots & a_{i-3} & g_{i-2}(z) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{b}_{i-2} & \bar{b}_{i-3} & \cdots & \bar{b}_0 & 0 & 0 & \cdots & a_0 & g_1(z) \\ 0 & 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_{i-1} & h_i(z) \\ \bar{a}_0 & 0 & \cdots & 0 & 0 & b_0 & \cdots & b_{i-2} & h_{i-1}(z) \\ \bar{a}_1 & \bar{a}_0 & \cdots & 0 & 0 & 0 & \cdots & b_{i-3} & h_{i-2}(z) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{a}_{i-2} & \bar{a}_{i-3} & \cdots & \bar{a}_0 & 0 & 0 & \cdots & b_0 & h_1(z) \end{vmatrix} \quad (8)$$

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$$\Delta_i(z) = \begin{vmatrix} \bar{b}_0 & 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_{i-2} & g_{i-1}(z) \\ \bar{b}_1 & \bar{b}_0 & \cdots & 0 & 0 & a_0 & \cdots & a_{i-3} & g_{i-2}(z) \\ \bar{b}_2 & \bar{b}_1 & \cdots & 0 & 0 & 0 & \cdots & a_{i-4} & g_{i-3}(z) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{b}_{i-1} & \bar{b}_{i-2} & \cdots & \bar{b}_0 & 0 & 0 & \cdots & 0 & g_0(z) \\ \bar{a}_0 & 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_{i-2} & h_{i-1}(z) \\ \bar{a}_1 & \bar{a}_0 & \cdots & 0 & 0 & b_0 & \cdots & b_{i-3} & h_{i-2}(z) \\ \bar{a}_2 & \bar{a}_1 & \cdots & 0 & 0 & 0 & \cdots & b_{i-4} & h_{i-3}(z) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{a}_{i-1} & \bar{a}_{i-2} & \cdots & \bar{a}_0 & 0 & 0 & \cdots & 0 & h_0(z) \end{vmatrix} \quad (9)$$

A Toeplitz reformulation

Following [Toeplitz, 1911] Schur associated matrices to every **power series** as follows. Let **a** and **b** be the power series defined as

$$a(z) = \sum_{k=0}^{\infty} a_k z^k, \quad b(z) = \sum_{k=0}^{\infty} b_k z^k. \quad (10)$$

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Then **infinite upper triangular Toeplitz matrices A and B** associated to a and b are defined by

$$A = \begin{pmatrix} a_0 & a_1 & \cdots \\ & a_0 & \ddots \\ & & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} b_0 & b_1 & \cdots \\ & b_0 & \ddots \\ & & \ddots \end{pmatrix}. \quad (11)$$

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and **infinite Hermitean matrices \mathfrak{A} and \mathfrak{B}** associated to a and b are defined by $\mathfrak{A} = A^H A$ and $\mathfrak{B} = B^H B$.

The matrix counterpart

The Schur coefficients are given by

$$\gamma_i = s_i(0) \tag{12}$$

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Due to the underlying relations, the reflection coefficients satisfy

$$1 - |\gamma_i|^2 = \frac{\delta_{i-1}\delta_{i+1}}{\delta_i^2}, \quad \delta_0 \equiv 1, \quad \delta_{-1} \equiv \frac{1}{b_0^2}. \quad (13)$$

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Based on the commutativity of Toeplitz matrices Schur proved that the determinants δ_i are also the $i \times i$ **leading principal determinants** of the infinite Hermitean matrix (Hermitean form)

$$\mathfrak{H} = \mathfrak{B} - \mathfrak{A}. \quad (14)$$

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[Schur, 1917/1918] remarks on pages 217–218:

*“Der Übergang [...] entspricht also dem ersten Schritt bei der **Jacobischen Transformation** der Form $\mathfrak{H}(x_0, x_1, \dots, x_\nu)$.”*

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Schur's treatment reminds of [Toeplitz, 1907] who cites Jacobi, but remarks:

“Allgemein für Bilinearformen wird diese Transformation von Jacobi aufgestellt; für quadratische Formen wird sie schon von Lagrange und Gauß verwendet.”

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[Lev-Ari & Kailath, 1986] state “incorrectly” (i.e., simplified) that the mentioned “Jacobi Transformation” [Jacobi, 1857] is the nested computation of the LDLT decomposition of H (a finite section of \mathfrak{H}),

$$H = LDL^H, \quad (15)$$

with L **unit** diagonal lower triangular and D diagonal.

Jacobi transformation; modern style

In [Kailath & Sayed, 1999] the authors “correct” the decomposition and call “Schur’s” variant of Jacobi’s transformation the **Gauß-Schur reduction**.

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Actually, [Jacobi, 1857] computes an **LDMT** decomposition using $A^{(0)} = A$ and the iteration

$$A^{(i)} = A^{(i-1)} - \frac{a_{:i}^{(i-1)} a_{i:}^{(i-1)}}{a_{ii}^{(i-1)}}. \quad (17)$$

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Additionally, the resulting quantities are expressed (as usually those times) in terms of determinants.

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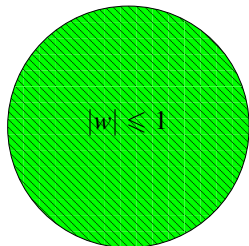
A generalized Schur algorithm

Cayley transform

The Cayley transform maps the open resp.
closed **half-plane** onto the open resp.
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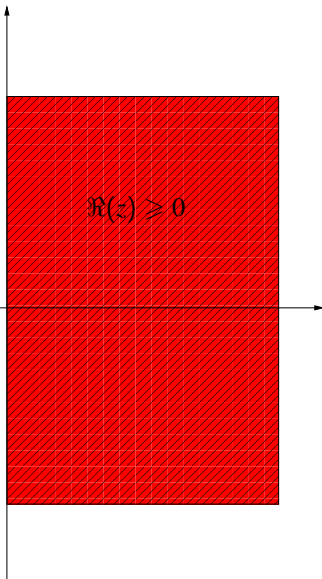
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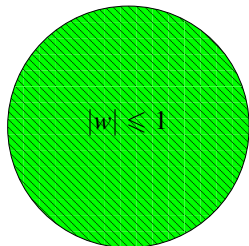
$$z = \frac{1-w}{1+w}$$

$$w = \frac{1-z}{1+z}$$



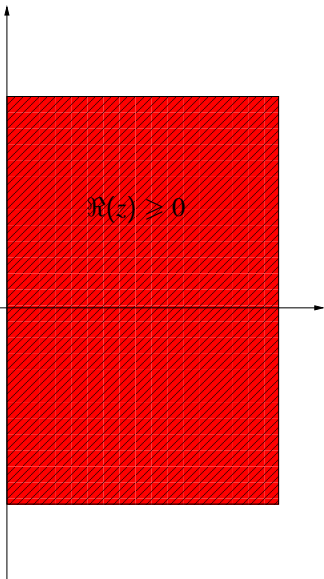
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This conformal map preserves analyticity.

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Carathéodory functions

Schur's investigations are closely related to Toeplitz' and Carathéodory's work on **functions with positive real part**,

$$c(z) = \sum_{k=0}^{\infty} c_k z^k, \quad \Re(c(z)) > 0, \quad |z| < 1. \quad (18)$$

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These functions are known as **Carathéodory functions**. A function is a Carathéodory function, if and only if "all" Hermitean Toeplitz matrices

$$T_m = \begin{pmatrix} \bar{c}_0 + c_0 & c_1 & \cdots & c_m \\ \bar{c}_1 & \bar{c}_0 + c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_1 \\ \bar{c}_m & \cdots & \bar{c}_1 & \bar{c}_0 + c_0 \end{pmatrix} \quad (19)$$

are positive definite, $\{T_m > 0\}_{m=0}^n$, $\det(T_m) = 0$, $m > n$.

Schur's proof

Carathéodory functions can be transformed to Schur functions via a Cayley transform that maps the positive real complex (right) half-plane onto the interior of the unit disc,

$$s(z) = \frac{1 - c(z)}{1 + c(z)}. \quad (20)$$

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If the Carathéodory function is given in rational form,

$$c(z) = \frac{a(z)}{b(z)} = \frac{\sum_{k=0}^{\infty} a_k z^k}{\sum_{k=0}^{\infty} b_k z^k}, \quad b_0 \neq 0, \quad (21)$$

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then the associated Schur function takes the form

$$s(z) = \frac{b(z) - a(z)}{b(z) + a(z)}. \quad (22)$$

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and taking $b(z) \equiv 1$ gives a multiple of the aforementioned Toeplitz matrix,

$$\mathfrak{H} = 2(B^H A + A^H B) = 2(I^H A + A^H I) \quad (25)$$

$$= 2(A + A^H) = 2(C + C^H). \quad (26)$$

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It turns out that the Schur algorithm (as a byproduct) **cheaply computes** the Cholesky decomposition.

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- Jacobi transformation

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Generalized Schur algorithms

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Schur's expansion

With

$$s_i(z) = \frac{a_i(z)}{b_i(z)}, \quad G_i(z) = \begin{pmatrix} b_i(z) & a_i(z) \end{pmatrix} \quad (29)$$

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Suppose we are given a positive definite finite section H of a symmetric infinite matrix

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Notation is changed slightly: Let $L(a)$ denote a **lower** triangular Toeplitz matrix with entries $a \in \mathbb{C}^n$ in the first column. Then

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The displacement can be described by shifting the entries of a and b , i.e., the elements of the power series a and b .

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Using the fact that Z is nilpotent and utilizing a Neumann series,

$$\text{vec}(H) = (I - Z \otimes Z)^{-1} \text{vec}(GJG) = \sum_{j=0}^{n-1} (Z \otimes Z)^j \text{vec}(GJG^H) \quad (37)$$

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and therefore we can **recover** H using the **generator** G ,

$$H = \sum_{j=0}^{n-1} Z^j GJG^H (Z^T)^j. \quad (38)$$

Displacement structure

Generators are unique up to post-multiplication by J -unitary matrices $\Theta \in \mathbb{C}$, defined by $\Theta J \Theta^H = J$, including **hyperbolic rotations** $\Theta(\gamma)$,

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Since H is assumed positive definite ($0 < h_{11} = |b^T e_1|^2 - |a^T e_1|^2$), we can always chose a generator G in “proper form”

$$G = \begin{pmatrix} b & a \end{pmatrix} = \begin{pmatrix} \star & \mathbf{0} \\ \star & \star \\ \vdots & \vdots \\ \star & \star \end{pmatrix}. \quad (41)$$

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The Cholesky decomposition of HPD H **can be computed** as follows. Set $H_1 = H$ and repeat for $i = 1, \dots, n$

$$H_{i+1} = H_i - c_i c_i^H, \quad (42)$$

where $c_{ii} = \sqrt{h_{ii}^{(i)}}$ and $c_i = H_i e_i / c_{ii}$.

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This style is not very economic and used **for demonstration only**.

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We have seen that a HPD Toeplitz matrix (scaled; now denoted by T),

$$T = \begin{pmatrix} 1 & \bar{t}_1 & \cdots & \bar{t}_n \\ t_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \bar{t}_1 \\ t_n & \cdots & t_1 & 1 \end{pmatrix} \quad (43)$$

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It is not hard to see that $G = \begin{pmatrix} 1 & 0 \\ t_1 & t_1 \\ \vdots & \vdots \\ t_n & t_n \end{pmatrix}$ is a **generator in proper form**.

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The **only portion** of

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Stripping of leading zero blocks in the matrix and the generator we can go through the steps and compute only the **nonzero** elements of the columns of the Cholesky factor.

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It turns out that we have recovered the Schur algorithm, since this is just another way of describing Schur's classical algorithm

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Cholesky decomposition

It turns out that we have recovered the Schur algorithm, since this is just another way of describing Schur's classical algorithm

$$s_{i+1}(z) = \frac{1}{z} \frac{s_i(z) - \gamma_i}{1 - \bar{\gamma}_i s_i(z)}, \quad \gamma_i = s_i(0), \quad (50)$$

for the case of rational functions

$$s_i(z) = \frac{a_i(z)}{b_i(z)}, \quad (51)$$

linearized in form of a coupled iteration

$$G_{i+1}(z) = G_i(z) \phi_i \begin{pmatrix} 1 & -\gamma_i \\ -\bar{\gamma}_i & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \quad \phi_i = \frac{1}{\sqrt{1 - |\gamma_i|^2}}, \quad (52)$$

with generators $G_i(z) = \begin{pmatrix} b_i(z) & a_i(z) \end{pmatrix}$.

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Start with a generator $G = \begin{pmatrix} b_1 & a_1 \end{pmatrix}$ of HPD Toeplitz T in proper form.

Iterate: for $i = 1, \dots, n - 1$:

$$c_i \leftarrow b_i$$

$$\gamma_i \leftarrow \frac{e_{i+1}^T a_i}{e_i^T b_i}$$

$$\begin{pmatrix} b_{i+1} & a_{i+1} \end{pmatrix} \leftarrow \begin{pmatrix} Zb_i & a_i \end{pmatrix} \frac{1}{\sqrt{1-|\gamma_i|^2}} \begin{pmatrix} 1 & -\gamma_i \\ -\bar{\gamma}_i & 1 \end{pmatrix}$$

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By previous considerations C iteratively defined by $Ce_i = c_i$ is the Cholesky factor of T .

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Let $R \in \mathbb{C}^{n \times n}$ be Hermitean, $R = R^H$. The **displacement** of R with respect to $F \in \mathbb{C}$ is defined by

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An example are symmetric real Toeplitz matrices T having displacement structure with respect to the shift matrix

$$F = Z = \begin{pmatrix} 0 & & & & \\ 1 & \ddots & & & \\ & \ddots & 0 & & \\ & & 1 & 0 & \\ & & & & \end{pmatrix}. \quad (55)$$

Displacement structure

Other types of displacement include the **Stein type displacement**

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Matrices with Stein type low displacement rank are termed **Toeplitz-like**, those with Sylvester type low displacement rank **Hankel-like**.

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The so-called **Pick matrix** A is defined by

$$A = \left(\frac{x_i x_j^H - y_i y_j^H}{1 - f_i f_j^H} \right)_{i,j=1}^n, \quad (59)$$

where $x_i \in \mathbb{C}^{1 \times p}$ and $y_i \in \mathbb{C}^{1 \times q}$ are complex row vectors and f_i are complex points inside the open unit disc.

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Such A has displacement rank $r = p + q$ with respect to

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since

$$A - FAF^H = \begin{pmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{pmatrix} \begin{pmatrix} I_p & O_{p,q} \\ O_{q,p} & -I_q \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{pmatrix}^H. \quad (61)$$

Displacement structure

A nonsymmetric example is given by a **Vandermonde matrix**

$$V = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^n \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^n \end{pmatrix}. \quad (62)$$

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$$V - FVZ^T = ee_1^T, \quad (64)$$

where e denotes the vector of all ones and e_1 is the first standard unit vector.

Displacement structure

Another example, this time with respect to a **Sylvester type displacement** is a **Cauchy matrix** C defined by

$$C = \begin{pmatrix} \frac{1}{x_1 - y_1} & \cdots & \frac{1}{x_1 - y_n} \\ \vdots & \ddots & \vdots \\ \frac{1}{x_n - y_1} & \cdots & \frac{1}{x_n - y_n} \end{pmatrix}. \quad (65)$$

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A well-known example of a Cauchy matrix is the famous **Hilbert matrix** H with entries

$$h_{ij} = \frac{1}{i + j - 1}. \quad (67)$$

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Recovering R

For simplicity we assume the Hermitean Stein displacement case

$$\nabla R = R - FRF^H = GJG^H, \quad J = J^H, \quad J^2 = I, \quad (68)$$

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The generalization can be based on some observations using a few related **block matrix decompositions**.

Fundamental properties

Displacement structure is preserved under inversion: there exists a full rank matrix $H \in \mathbb{C}^{r \times n}$ such that

$$R^{-1} - F^H R^{-1} F = H^H J H. \quad (72)$$

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Proof: The block matrix triangular decompositions

$$\begin{pmatrix} R & F \\ F^H & R^{-1} \end{pmatrix} = \begin{pmatrix} I & O \\ F^H R^{-1} & I \end{pmatrix} \begin{pmatrix} R & O \\ O & R^{-1} - F^H R^{-1} F \end{pmatrix} \begin{pmatrix} I & O \\ F^H R^{-1} & I \end{pmatrix}^H \quad (73)$$

$$= \begin{pmatrix} I & FR \\ O & I \end{pmatrix} \begin{pmatrix} R - FRF^H & O \\ O & R^{-1} \end{pmatrix} \begin{pmatrix} I & FR \\ O & I \end{pmatrix}^H \quad (74)$$

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show that by Sylvester's law of inertia

$$\text{inertia}(R^{-1} - F^H R^{-1} F) = \text{inertia}(R - FRF^H). \quad (75)$$

Fundamental properties

We partition

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} F_{12} & O \\ F_{21} & F_{22} \end{pmatrix}. \quad (76)$$

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By the first result

$$\text{rank}(S_{22} - F_{22}S_{22}F_{22}^H) = \text{rank}(S_{22}^{-1} - F_{22}^H S_{22}^{-1} F_{22}). \quad (79)$$

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$$\begin{pmatrix} FL & G \\ U & O \end{pmatrix} \Omega = \begin{pmatrix} L & O \\ F^H U & H^H \end{pmatrix}, \quad \Omega(D^{-1} \oplus J)\Omega = (D^{-1} \oplus J). \quad (81)$$

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We restrict interest to the **leading block row**.

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Schur algorithm; modern form

- Reformulation of Schur's expansion

- Displacement structure

- Cholesky decomposition

- The Schur algorithm

Generalized Schur algorithms

- Displacement structure

- Fundamental properties

- A generalized Schur algorithm**

Sketch of an algorithm

The key array equation looks as

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$$(FL \ G) \Omega_0 = \begin{pmatrix} l_0 & o^T & o^T \\ & F_1 L_1 & G_1 \end{pmatrix}. \quad (85)$$

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We suppose this is to be correct and proceed to compute

$$(FL \ G) \Omega_0 \Omega_1 \cdots \Omega_{i-1} = \begin{pmatrix} \hat{L}_i & O & O \\ \tilde{L}_i & F_i L_i & G_i \end{pmatrix}. \quad (87)$$

Sketch of an algorithm

It remains to prove that in the i th step with $R_i = L_i D_i^{-1} L_i^H$,

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It follows immediately from equating second block rows,

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that

$$F_i (L_i D_i^{-1} L_i^H) F_i^H + G_i J G_i^H = (L_i D_i^{-1} L_i^H). \quad (90)$$

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Thus, the displacement structure of the Schur complements has been verified.

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The next column of L is (Jacobi-style scaling) given by $l_i = R_i e_1$ and thus we have to “solve” the linear system

$$(I_{n-i} - \bar{f}_i F_i) R_i e_1 = (I_{n-i} - \bar{f}_i F_i) l_i = G_i J G_i^H e_1. \quad (92)$$

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Afterwards we compute the new generator with a $d_i^{-1} \oplus J$ -unitary transformation Ω_i (the leading columns remain the same),

$$\begin{pmatrix} l_i & o^T \\ & G_{i+1} \end{pmatrix} = (F_i l_i \quad G_i) \Omega_i. \quad (93)$$

Sketch of an algorithm

Not surprisingly it turns out the d_i^{-1} part of the $d_i^{-1} \oplus J$ -unitary transformation Ω_i ,

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The transformation takes the form

$$\begin{pmatrix} o^T \\ G_{i+1} \end{pmatrix} = \left(G_i + (\Phi_i - I_{n-i}) G_i \frac{J g_i^H g_i}{g_i J g_i^H} \right) \Theta_i, \quad (95)$$

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is the so-called Blaschke-Potapov matrix, $g_i = e_1^T G_i$ is the leading row of G_i and Θ_i is a J -unitary matrix chosen to introduce leading zeros.

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Set $G_0 = G$, $i = 0$ and iterate the following: Set $g_i = e_1^T G_i$.

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$$d_i = \frac{g_i J g_i^H}{1 - f_i \bar{f}_i} = e_1^T l_i. \quad (98)$$

Compute a new G_{i+1} using a J -unitary matrix Θ_i ,

$$\begin{pmatrix} o^T \\ G_{i+1} \end{pmatrix} = \begin{pmatrix} G_i + (\Phi_i - I_{n-i}) G_i \frac{J g_i^H g_i}{g_i J g_i^H} \end{pmatrix} \Theta_i, \quad (99)$$

where $\Phi_i = (F_i - f_i I_{n-i})(I_{n-i} - \bar{f}_i F_i)^{-1}$.



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