On Generalized Schur Algorithms

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Outline

Classification and normal forms of functions

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- Schur functions
- Jacobi transformation
- Cayley transform
- Carathéodory functions
- Matrix decomposition

Schur algorithm; modern form

- Reformulation of Schur's expansion
- Displacement structure
- Cholesky decomposition
- The Schur algorithm

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$$s(z) = \sum_{k=0}^{\infty} \mathfrak{s}_k z^k, \qquad |s(z)| \leqslant 1, \quad |z| < 1. \tag{1}$$

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It turns out that the coefficients $\mathfrak{s}_k$ of the power series are not that useful when investigating Schur functions. Instead, the so-called Schur coefficients better describe their properties, and thus, play the dominant rôle.

Expansion of Schur functions

Schur used that for $\alpha \in \mathbb{C}$ with $|\alpha| < 1$ the "linear" transformation (Blaschke factor, no normalization; Moebius transformation)

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$$s_{i+1}(z) = \frac{1}{z} \frac{s_i(z) - \gamma_i}{1 - \overline{\gamma}_i s_i(z)}, \qquad \gamma_i = s_i(0). \tag{4}$$

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Then all $\{s_i\}_{i=0}^m$ satisfy $|s_i(z)| \leqslant 1$. The constants γ_i are the aforementioned Schur coefficients or reflection coefficients.

Rational expansion of Schur functions

Schur developed recursions for the expansion by writing the function s in terms of a rational function,

$$s(z) = \frac{a(z)}{b(z)} = \frac{\sum_{k=0}^{\infty} a_k z^k}{\sum_{k=0}^{\infty} b_k z^k}, \qquad b_0 \neq 0. \tag{5}$$

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where he additionally defined analytic functions

$$g_i(z) = \sum_{k=0}^{\infty} a_{i+k} z^k \quad \text{and} \quad h_i(z) = \sum_{k=0}^{\infty} b_{i+k} z^k \tag{7}$$

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and gave determinal expressions for $D_i(z)$ and $\Delta_i(z)$ (the next two pages . . .)

Explicit solution of the recursion

(remember that $g_i(z) = \sum_{k=0}^{\infty} a_{i+k} z^k$ and $h_i(z) = \sum_{k=0}^{\infty} b_{i+k} z^k$)

$$D_i(z) = \begin{vmatrix} 0 & 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_{i-1} & g_i(z) \\ \overline{b}_0 & 0 & \cdots & 0 & 0 & a_0 & \cdots & a_{i-2} & g_{i-1}(z) \\ \overline{b}_1 & \overline{b}_0 & \cdots & 0 & 0 & 0 & \cdots & a_{i-3} & g_{i-2}(z) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \overline{b}_{i-2} & \overline{b}_{i-3} & \cdots & \overline{b}_0 & 0 & 0 & \cdots & a_0 & g_1(z) \\ 0 & 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_{i-1} & h_i(z) \\ \overline{a}_0 & 0 & \cdots & 0 & 0 & b_0 & \cdots & b_{i-2} & h_{i-1}(z) \\ \overline{a}_1 & \overline{a}_0 & \cdots & 0 & 0 & 0 & \cdots & b_{i-3} & h_{i-2}(z) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \overline{a}_{i-2} & \overline{a}_{i-3} & \cdots & \overline{a}_0 & 0 & 0 & \cdots & b_0 & h_1(z) \end{vmatrix} \qquad (8)$$

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A Toeplitz reformulation

Following [Toeplitz, 1911] Schur associated matrices to every power series as follows. Let a and b be the power series defined as

$$a(z) = \sum_{k=0}^{\infty} a_k z^k, \quad b(z) = \sum_{k=0}^{\infty} b_k z^k. \tag{10}$$

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Then infinite upper triangular Toeplitz matrices A and B associated to a and b are defined by

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and infinite Hermitean matrices $\mathfrak{A}$ and $\mathfrak{B}$ associated to a and b are defined by $\mathfrak{A} = A^H A$ and $\mathfrak{B} = B^H B$.

The matrix counterpart

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$$\gamma_i = s_i(0) \tag{12}$$

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$$1 - |\gamma_i|^2 = \frac{\delta_{i-1}\delta_{i+1}}{\delta_i^2}, \quad \delta_0 \equiv 1,\ \delta_{-1} \equiv \frac{1}{b_0^2}. \tag{13}$$

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Based on the commutativity of Toeplitz matrices Schur proved that the determinants δ_i are also the $i \times i$ leading principal determinants of the infinite Hermitean matrix (Hermitean form)

$$\mathfrak{H} = \mathfrak{B} - \mathfrak{A}. \tag{14}$$

Outline

Jacobi transformation

[Schur, 1917/1918] remarks on pages 217–218:

> *"Der Übergang [..] entspricht also dem ersten Schritt bei der Jacobischen Transformation der Form* $\mathfrak{H}(x_0, x_1, \ldots, x_\nu)$*."*

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Schur’s treatment reminds of [Toeplitz, 1907] who cites Jacobi, but remarks:

> *“Allgemein für Bilinearformen wird diese Transformation von Jacobi aufgestellt; für quadratische Formen wird sie schon von Lagrange und Gauß verwendet.”*

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[Lev-Ari & Kailath, 1986] state "incorrectly" (i.e., simplified) that the mentioned "Jacobi Transformation" [Jacobi, 1857] is the nested computation of the LDLT decomposition of H (a finite section of $\mathfrak{H}$),

$$H = LDL^H, \tag{15}$$

with L unit diagonal lower triangular and D diagonal.

Jacobi transformation; modern style

In [Kailath & Sayed, 1999] the authors “correct” the decomposition and call “Schur’s” variant of Jacobi’s transformation the Gauß-Schur reduction.

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Additionally, the resulting quantities are expressed (as usually those times) in terms of determinants.

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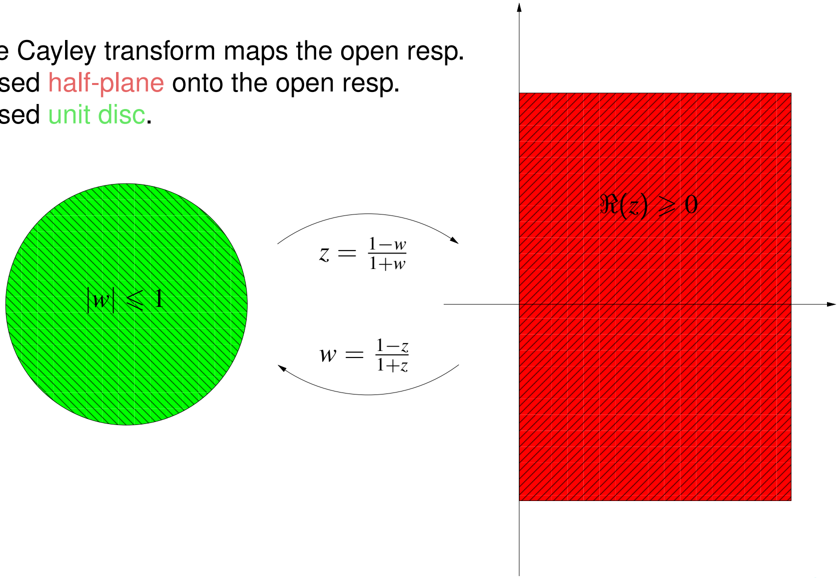

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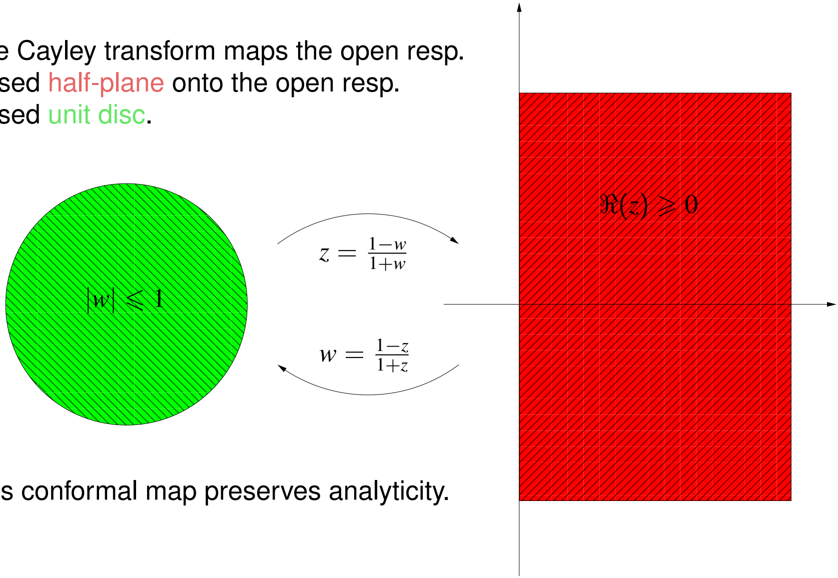

This conformal map preserves analyticity.

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Carathéodory functions

Schur's investigations are closely related to Toeplitz' and Carathéodory's work on functions with positive real part,

$$c(z) = \sum_{k=0}^{\infty} c_k z^k, \qquad \Re(c(z)) > 0, \quad |z| < 1. \tag{18}$$

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$$T_m = \begin{pmatrix} \overline{c}_0 + c_0 & c_1 & \cdots & c_m \\ \overline{c}_1 & \overline{c}_0 + c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_1 \\ \overline{c}_m & \cdots & \overline{c}_1 & \overline{c}_0 + c_0 \end{pmatrix} \tag{19}$$

are positive definite, $\{T_m > 0\}_{m=0}^n$, $\det(T_m) = 0$, $m > n$.

Schur's proof

Carathéodory functions can be transformed to Schur functions via a Cayley transform that maps the positive real complex (right) half-plane onto the interior of the unit disc,

$$s(z) = \frac{1 - c(z)}{1 + c(z)}. \tag{20}$$

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If the Carathéodory function is given in rational form,

$$c(z) = \frac{a(z)}{b(z)} = \frac{\sum_{k=0}^{\infty} a_k z^k}{\sum_{k=0}^{\infty} b_k z^k}, \qquad b_0 \neq 0, \tag{21}$$

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then the associated Schur function takes the form

$$s(z) = \frac{b(z) - a(z)}{b(z) + a(z)}. \tag{22}$$

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and taking $b(z) \equiv 1$ gives a multiple of the aforementioned Toeplitz matrix,

$$\mathfrak{H} = 2(B^H A + A^H B) = 2(I^H A + A^H I) \tag{25}$$

$$= 2(A + A^H) = 2(C + C^H). \tag{26}$$

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It turns out that the Schur algorithm (as a byproduct) cheaply computes the Cholesky decomposition.

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$$G_{i+1}(z) = G_i(z)\ \phi_i \begin{pmatrix} 1 & -\gamma_i \\ -\overline{\gamma}_i & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \quad \phi_i = \text{arbitrary.} \tag{30}$$

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Reversing the approach

Suppose we are given a positive definite finite section H of a symmetric infinite matrix

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Notation is changed slightly: Let $L(a)$ denote a lower triangular Toeplitz matrix with entries $a \in \mathbb{C}^n$ in the first column. Then

$$H = L(b)L(b)^H - L(a)L(a)^H \tag{34}$$

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The displacement can be described by shifting the entries of a and b, i.e., the elements of the power series a and b.

Outline

Displacement structure

Let $Z = \mathrm{diag}(\text{ones}(n-1,1),-1)$ denote the displacement by one position. Then

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Using the fact that Z is nilpotent and utilizing a Neumann series,

$$\mathrm{vec}(H) = (I - Z \otimes Z)^{-1}\mathrm{vec}(GJG) = \sum_{j=0}^{n-1} (Z \otimes Z)^j \mathrm{vec}(GJG^H) \tag{37}$$

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and therefore we can recover H using the generator G,

$$H = \sum_{j=0}^{n-1} Z^j GJG^H (Z^T)^j. \tag{38}$$

Displacement structure

Generators are unique up to post-multiplication by J-unitary matrices $\Theta \in \mathbb{C}$, defined by $\Theta J \Theta^H = J$, including hyperbolic rotations $\Theta(\gamma)$,

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Since H is assumed positive definite ($0 < h_{11} = |b^T e_1|^2 - |a^T e_1|^2$), we can always chose a generator G in “proper form”

$$G = \begin{pmatrix} b & a \end{pmatrix} = \begin{pmatrix} \star & {\color{red}0} \\ \star & \star \\ \vdots & \vdots \\ \star & \star \end{pmatrix}. \tag{41}$$

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Cholesky decomposition

The generator can be used to compactly ($2n$ entries) represent a HPD Toeplitz matrix (n^2 entries). It is slightly more expensive than parameterization by first row or column (n entries) but enables a cheap triangular decomposition.

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The Cholesky decomposition of HPD H can be computed as follows. Set $H_1 = H$ and repeat for $i = 1, \dots, n$

$$H_{i+1} = H_i - c_i c_i^H, \tag{42}$$

where $c_{ii} = \sqrt{h_{ii}^{(i)}}$ and $c_i = H_i e_i / c_{ii}$.

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This style is not very economic and used for demonstration only.

Cholesky decomposition

We have seen that a HPD Toeplitz matrix (scaled; now denoted by T),

$$T = \begin{pmatrix} 1 & \bar{t}_1 & \cdots & \bar{t}_n \\ t_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \bar{t}_1 \\ t_n & \cdots & t_1 & 1 \end{pmatrix} \tag{43}$$

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Stripping of leading zero blocks in the matrix and the generator we can go through the steps and compute only the nonzero elements of the columns of the Cholesky factor.

Cholesky decomposition

It turns out that we have recovered the Schur algorithm, since this is just another way of describing Schur's classical algorithm

$$s_{i+1}(z) = \frac{1}{z}\frac{s_i(z)-\gamma_i}{1-\overline{\gamma}_i s_i(z)}, \qquad \gamma_i = s_i(0). \tag{50}$$

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with generators $G_i(z) = \begin{pmatrix} b_i(z) & a_i(z) \end{pmatrix}$.

Outline

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Stated in terms of vectors $a_i, b_i \in \mathbb{C}^n$ representing the coefficients of the respective power series and using the shift operator Z in place of multiplication by z we have shown that the Schur algorithm takes the following form:

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Iterate: for $i = 1, \ldots, n-1$:

$$c_i \leftarrow b_i$$

$$\gamma_i \leftarrow \frac{e_{i+1}^T a_i}{e_i^T b_i}$$

$$\begin{pmatrix} b_{i+1} & a_{i+1} \end{pmatrix} \leftarrow \begin{pmatrix} Zb_i & a_i \end{pmatrix} \frac{1}{\sqrt{1-|\gamma_i|^2}} \begin{pmatrix} 1 & -\gamma_i \\ -\overline{\gamma}_i & 1 \end{pmatrix}$$

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By previous considerations C iteratively defined by $Ce_i = c_i$ is the Cholesky factor of T.

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The use of augmented & composed systems is used to develop variants to compute:

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Cholesky decomposition: this is the “standard” Schur algorithm

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Outline

Displacement structure

Let $R \in \mathbb{C}^{n\times n}$ be Hermitean, $R = R^H$. The displacement of R with respect to $F \in \mathbb{C}$ is defined by

$$\nabla_F R = R - FRF^H. \tag{53}$$

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An example are symmetric real Toeplitz matrices T having displacement structure with respect to the shift matrix

$$F = Z = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{pmatrix}. \tag{55}$$

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Other types of displacement include the Stein type displacement

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Matrices with Stein type low displacement rank are termed Toeplitz-like, those with Sylvester type low displacement rank Hankel-like.

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The so-called Pick matrix A is defined by

$$A = \left(\frac{x_i x_j^H - y_i y_j^H}{1 - f_i f_j^H} \right)_{i,j=1}^{n}, \tag{59}$$

where $x_i \in \mathbb{C}^{1\times p}$ and $y_i \in \mathbb{C}^{1\times q}$ are complex row vectors and f_i are complex points inside the open unit disc.

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$$A - FAF^H = \begin{pmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{pmatrix} \begin{pmatrix} I_p & O_{p,q} \\ O_{q,p} & -I_q \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{pmatrix}^H. \tag{61}$$

Displacement structure

A nonsymmetric example is given by a Vandermonde matrix

$$V = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^n \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^n \end{pmatrix}. \tag{62}$$

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and the shift matrix Z introduced before, since

$$V - FVZ^T = ee_1^T, \tag{64}$$

where e denotes the vector of all ones and e_1 is the first standard unit vector.

Displacement structure

Another example, this time with respect to a Sylvester type displacement is a Cauchy matrix C defined by

$$C = \begin{pmatrix} \frac{1}{x_1 - y_1} & \cdots & \frac{1}{x_1 - y_n} \\ \vdots & \ddots & \vdots \\ \frac{1}{x_n - y_1} & \cdots & \frac{1}{x_n - y_n} \end{pmatrix}. \tag{65}$$

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A well-known example of a Cauchy matrix is the famous Hilbert matrix H with entries

$$h_{ij} = \frac{1}{i + j - 1}. \tag{67}$$

Outline

Recovering R

For simplicity we assume the Hermitean Stein displacement case

$$\nabla R = R - FRF^H = GJG^H, \qquad J = J^H, \quad J^2 = I, \tag{68}$$

where $G \in \mathbb{C}^{n\times r}$ has full rank.

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The generalization can be based on some observations using a few related block matrix decompositions.

Fundamental properties

Displacement structure is preserved under inversion: there exists a full rank matrix $H \in \mathbb{C}^{r\times n}$ such that

$$R^{-1} - F^H R^{-1} F = H^H J H. \tag{72}$$

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Proof: The block matrix triangular decompositions

$$\begin{pmatrix} R & F \\ F^H & R^{-1} \end{pmatrix} = \begin{pmatrix} I & O \\ F^H R^{-1} & I \end{pmatrix} \begin{pmatrix} R & O \\ O & R^{-1} - F^H R^{-1} F \end{pmatrix} \begin{pmatrix} I & O \\ F^H R^{-1} & I \end{pmatrix}^H \tag{73}$$

$$= \begin{pmatrix} I & FR \\ O & I \end{pmatrix} \begin{pmatrix} R - FRF^H & O \\ O & R^{-1} \end{pmatrix} \begin{pmatrix} I & FR \\ O & I \end{pmatrix}^H \tag{74}$$

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show that by Sylvester's law of inertia

$$\text{inertia}(R^{-1} - F^H R^{-1} F) = \text{inertia}(R - FRF^H). \tag{75}$$

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We partition

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} F_{12} & O \\ F_{21} & F_{22} \end{pmatrix}. \tag{76}$$

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By the first result

$$\operatorname{rank}(S_{22} - F_{22}S_{22}F_{22}^H) = \operatorname{rank}(S_{22}^{-1} - F_{22}^H S_{22}^{-1} F_{22}). \tag{79}$$

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We now use the Jacobi-style decompositions

$$R = LD^{-1}L^H, \quad U \equiv L^{-H}D, \quad R^{-1} = UD^{-1}U^H. \tag{80}$$

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$$\begin{pmatrix} FL & G \\ U & O \end{pmatrix} \Omega = \begin{pmatrix} L & O \\ F^H U & H^H \end{pmatrix}, \qquad \Omega(D^{-1} \oplus J)\Omega = (D^{-1} \oplus J). \tag{81}$$

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We restrict interest to the leading block row.

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Sketch of an algorithm

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The algorithm works recursively. We use a $D^{-1} \oplus J$-unitary matrix Ω_0 to obtain a partial triangularization like

$$\begin{pmatrix} FL & G \end{pmatrix} \Omega_0 = \begin{pmatrix} l_0 & o^T & o^T \\ & F_1 L_1 & G_1 \end{pmatrix}. \tag{85}$$

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We suppose this is to be correct and proceed to compute

$$\begin{pmatrix} FL & G \end{pmatrix} \Omega_0 \Omega_1 \cdots \Omega_{i-1} = \begin{pmatrix} \hat{L}_i & O & O \\ \tilde{L}_i & F_i L_i & G_i \end{pmatrix}. \tag{87}$$

Sketch of an algorithm

It remains to prove that in the ith step with $R_i = L_i D_i^{-1} L_i^H$,

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It follows immediately from equating second block rows,

$$\begin{aligned}\begin{pmatrix}\tilde{L}_i & F_i L_i & G_i\end{pmatrix}\left(\hat{D}_i^{-1} \oplus D_i^{-1} \oplus J\right)\begin{pmatrix}\tilde{L}_i & F_i L_i & G_i\end{pmatrix}^H &= \\ \begin{pmatrix}\tilde{L}_i & L_i & O\end{pmatrix}\left(\hat{D}_i^{-1} \oplus D_i^{-1} \oplus J\right)\begin{pmatrix}\tilde{L}_i & L_i & O\end{pmatrix}^H&,\end{aligned} \tag{89}$$

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that

$$F_i (L_i D_i^{-1} L_i^H) F_i^H + G_i J G_i^H = (L_i D_i^{-1} L_i^H). \tag{90}$$

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holds true.

It follows immediately from equating second block rows,

$$\begin{aligned} \begin{pmatrix} \tilde{L}_i & F_i L_i & G_i \end{pmatrix} \left(\hat{D}_i^{-1} \oplus D_i^{-1} \oplus J\right) \begin{pmatrix} \tilde{L}_i & F_i L_i & G_i \end{pmatrix}^H &= \\ \begin{pmatrix} \tilde{L}_i & L_i & O \end{pmatrix} \left(\hat{D}_i^{-1} \oplus D_i^{-1} \oplus J\right) \begin{pmatrix} \tilde{L}_i & L_i & O \end{pmatrix}^H&, \end{aligned} \tag{89}$$

that

$$F_i (L_i D_i^{-1} L_i^H) F_i^H + G_i J G_i^H = (L_i D_i^{-1} L_i^H). \tag{90}$$

Thus, the displacement structure of the Schur complements has been verified.

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The next column of L is (Jacobi-style scaling) given by $l_i = R_i e_1$ and thus we have to "solve" the linear system

$$(I_{n-i} - \bar{f}_i F_i) R_i e_1 = (I_{n-i} - \bar{f}_i F_i) l_i = G_i J G_i^H e_1. \tag{92}$$

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Afterwards we compute the new generator with a $d_i^{-1} \oplus J$-unitary transformation Ω_i (the leading columns remain the same),

$$\begin{pmatrix} l_i & o^T \\ & G_{i+1} \end{pmatrix} = \begin{pmatrix} F_i l_i & G_i \end{pmatrix} \Omega_i. \tag{93}$$

Sketch of an algorithm

Not surprisingly it turns out the d_i^{-1} part of the $d_i^{-1} \oplus J$-unitary transformation Ω_i,

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The transformation takes the form

$$\begin{pmatrix} o^T \\ G_{i+1} \end{pmatrix} = \left(G_i + (\Phi_i - I_{n-i}) G_i \frac{J g_i^H g_i}{g_i J g_i^H} \right) \Theta_i, \tag{95}$$

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is the so-called Blaschke-Potapov matrix, $g_i = e_1^T G_i$ is the leading row of G_i and Θ_i is a J-unitary matrix chosen to introduce leading zeros.

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$$d_i = \frac{g_i J g_i^H}{1 - f_i \overline{f}_i} = e_1^T l_i. \tag{98}$$

Compute a new G_{i+1} using a J-unitary matrix Θ_i,

$$\begin{pmatrix} o^T \\ G_{i+1} \end{pmatrix} = \left(G_i + (\Phi_i - I_{n-i}) G_i \frac{J g_i^H g_i}{g_i J g_i^H} \right) \Theta_i, \tag{99}$$

where $\Phi_i = (F_i - f_i I_{n-i})(I_{n-i} - \overline{f}_i F_i)^{-1}$.

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